

Determining the preferred directions of magnetisation in cubic crystals using symmetric polynomial inequalities

Fabian Samad^{1,2*} and Olav Hellwig^{1,2}

¹ Institute of Ion Beam Physics and Materials Research, Helmholtz-Zentrum Dresden-Rossendorf, Bautzner Landstraße 400, 01328 Dresden, Germany

² Institute of Physics, University of Technology Chemnitz, Reichenhainer Str. 70, 09126 Chemnitz, Germany

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Abstract. For a magnetic material, the easy and hard magnetic axes describe the directions of favourable respectively unfavourable alignment of the magnetisation. In this article, we describe how to determine these axes for cubic magnetic crystals. Usually it is assumed without further reasoning that they coincide with some principal symmetry directions of the crystal [Bozorth, Phys. Rev. **50**, 1076–1081 (1936)], which is however invalid in general. In contrast, we present a full and elementary analysis using symmetric polynomial inequalities, which are well suited to the symmetry of the problem.

Keywords: Anisotropy / cubic magnetic anisotropy / symmetric polynomial inequalities / symmetry / cubic crystals

1 Introduction

In crystalline magnetic materials, due to the interaction of the electric crystal field with the orbitals of the electrons relevant for the magnetism, together with spin-orbit coupling, the so-called magnetocrystalline anisotropy arises [1]. Namely, the Gibbs free energy density ε of the system depends on the direction of magnetisation in the crystal. The directions corresponding to the global maximum/minimum in ε are denoted hard/easy axes. In the absence of an external magnetic field (at remanence), the magnetisation is usually aligned along an easy axis of the system. Conversely, it requires a large external magnetic field to align the magnetisation along a hard axis. Thus, knowledge of the easy and hard axes is of great practical importance when utilising magnetic materials.

Cubic crystalline materials constitute an important class, to which e.g. the magnetic metals Fe and Ni belong. Figure 1a illustrates a body centred cubic (bcc) crystal unit cell, with the magnetisation vector being drawn in one of the $\langle 111 \rangle$ directions (note that, by convention, $\langle 111 \rangle$ represents all eight specific directions $[111], [\bar{1}\bar{1}\bar{1}]$ etc.). While from symmetry it is clear that in the cubic crystal all of the $\langle 111 \rangle$ directions are energetically equivalent, we *a priori* do not know if it corresponds to a global extremum of the free energy, e.g. if it is an easy axis and therefore the preferred magnetisation direction at remanence. The same

holds true for the other principal symmetry directions of the cube, *viz.* $\langle 110 \rangle$ and $\langle 100 \rangle$. Surprisingly, though, in early analyses of cubic magnetocrystalline anisotropy in the 1930's [2], it was ad hoc assumed that the easy and hard axes are always directed along these cubic symmetry directions. We found only one article from 1964, almost 30 years after reference [2], where this gap was recognised and a rigorous analysis was executed [3]. In fact, when including higher order terms (which can be of practical physical relevance [4]), an energy surface of cubic symmetry can occur, where global maxima or minima are *not* attained at symmetry directions [4], see Figure 1b. When considering orders ≥ 18 , even cubic symmetric polynomials with no global minimum in the $\{100\}$ or $\{110\}$ planes exist, e.g. $\varepsilon = -(\alpha_1\alpha_2\alpha_3(\alpha_1^2 - \alpha_2^2)(\alpha_2^2 - \alpha_3^2)(\alpha_3^2 - \alpha_1^2))^2$ [5] (where the α_i are the direction cosines of the magnetisation, see the method-section). In this context, we want to stress that it is pivotal to not only understand which implications the cubic symmetry has (namely that ε must exhibit cubic symmetry¹), but also which implications it does not have, here particularly concerning the extrema of ε . In an article by Waterhouse titled “Do symmetric problems have symmetric solutions?” [8] it is shown that in most cases, given a symmetric function under a symmetric constraint, there is a *local* extremum (or saddle

¹ That the physical properties have (at least) cubic symmetry also follows from the famous Curie-principle [6] (also called Neumann-Minnigerode-Curie principle, see Ref. [7])

* e-mail: fabian.samad@physik.tu-chemnitz.de

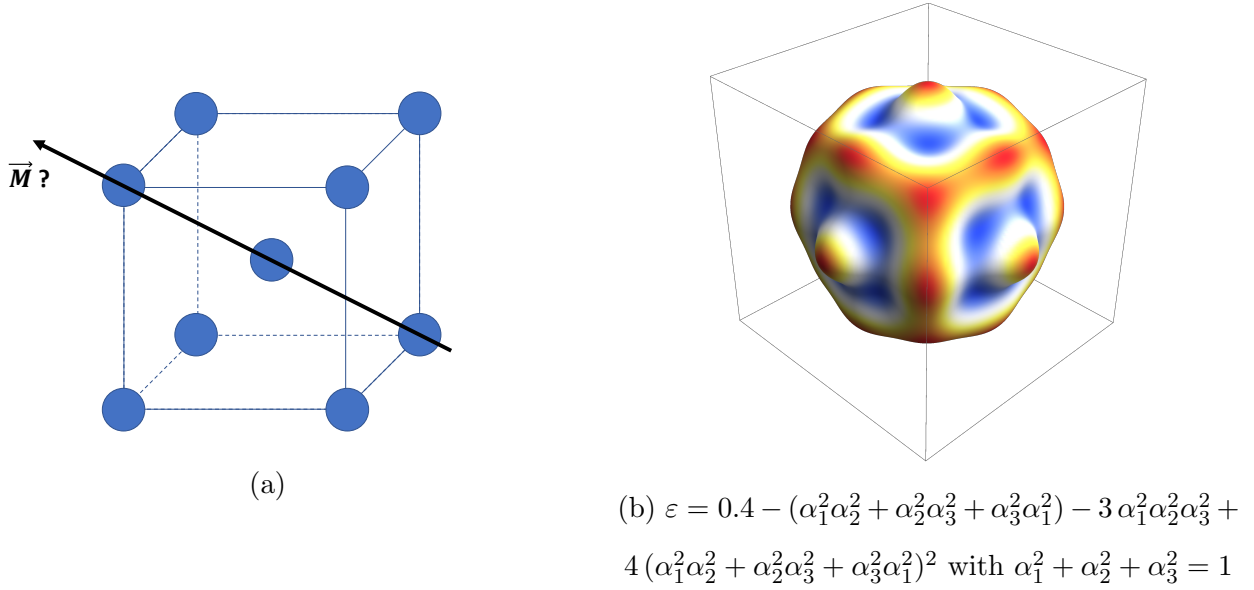


Fig. 1. Left: Schematic of a bcc crystal unit cell, with the overall magnetisation vector being oriented along one of the $\langle 111 \rangle$ directions, corresponding to a possible easy axis. Right: Spherical plot of a cubic anisotropy energy density ε considering terms up to eighth order. The magnitude of ε is represented by the distance from the cube centre and the colour code. The global minima (blue colour) remote from cubic high symmetry directions (such as in [4]) are clearly visible.

point) attained at the symmetry directions. This is sometimes denoted as extended Purkiss principle, named after Purkiss (1842–1865) being one of the first stating such a symmetry principle [8]. It does not, however, give information on possible global extrema away from the symmetry directions.²

Strikingly, both introductory as well as advanced lectures and textbooks on condensed matter physics or magnetism usually skip an analysis of ε , instead only the resulting easy and hard axes are presented [1,9,10]. Given the importance of cubic crystalline magnetic materials, this constitutes a crucial gap, even more so as the result is sometimes misunderstood as “immediately” following from symmetry. To resolve this, we give a full analysis using elementary methods, namely symmetric polynomial inequalities, which fit the problem’s symmetry and hence constitute a direct method for obtaining the easy and hard magnetic axes of cubic crystals. Given the ubiquity of symmetric polynomials, e.g. resulting from a Taylor expansion of a symmetric function, the method presented here can also be effectively applied to other akin problems (such as molecule rotations in fields of cubic symmetry [11,12]).

2 Method

We want to find the directions of magnetisation, for which the free energy density ε corresponding to the magnetocrystalline anisotropy is extremal. For cubic ferromagnetic materials, usually [10] it is enough to consider

ε up to sixth order in the directional cosines α_i of the magnetisation vector $(x_1, x_2, x_3)^T$, where $(\alpha_1, \alpha_2, \alpha_3) = (x_1, x_2, x_3)/r = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ with $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Employing the cubic symmetry (and omitting the isotropic zeroth order term), one finds [2,14]

$$\varepsilon = K_1 (\alpha_1^2\alpha_2^2 + \alpha_2^2\alpha_3^2 + \alpha_3^2\alpha_1^2) + K_2 \alpha_1^2\alpha_2^2\alpha_3^2, \quad (1)$$

with the constraint $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ and some anisotropy constants K_1, K_2 depending on the material. Alternatively, an expansion of the anisotropy functional in orthonormal spherical harmonics can be advantageous [5,15].

In principle, such a problem can be analysed using differential calculus [3,12]. However, we would like to note that the extrema and boundaries of physical systems might be examined much more effectively using adequate mathematical inequalities. Consider for example the arithmetic mean (AM) – geometric mean (GM) inequality, which states that the AM of non-negative numbers is not smaller than their GM, with equality if and only if all numbers are equal. This inequality can be effectively applied in a vast number of situations, given that terms with the structure of an AM (basically a sum) or GM (basically a product under the root) are ubiquitous [16]. In general, inequalities have the advantage of being applicable in situations where the functions are not differentiable [17] and they directly yield functions as bounds (such as the very “AM $\geq$ GM”). Additionally, global extrema are readily obtained instead of local extrema, which follow from differential calculus. This is important in our case, as the easy and hard magnetic axes correspond to global energy extrema.

²However, a working approach can be to compare the number of local extrema from the extended Purkiss principle with the number of zeros of the gradient of ε (with the constraint on the magnetisation incorporated), which can be estimated by Bézout’s theorem [13].

As indicated in the introduction of this article, we will perform our analysis using suitable inequalities corresponding to the symmetry of the free energy density ε . There are different ways to do this in practice. In the literature, similar symmetric polynomials have been investigated using geometrically motivated or other derived inequalities [18,19], or by deriving criteria for their positivity [20–22]. Here, we want to demonstrate a simple solution by directly applying two elementary mathematical inequalities, MacLaurin's inequality and Schur's inequality. In the following, these inequalities shall be formulated, while the proofs are given in the appendix. First, we establish the definition of elementary symmetric polynomials s_k . Their name reflects the surprising fact that every symmetric polynomial can be written as a polynomial in s_k , so that they can be viewed as building blocks of symmetric polynomials [23].

For n variables $x_1, \dots, x_n$ we define the elementary symmetric polynomials s_k and the weighted elementary symmetric polynomials d_k by

$$s_k = \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} \cdots x_{i_k}, \quad d_k = \frac{s_k}{\binom{n}{k}} \quad (2)$$

Particularly, for three variables x, y, z , we have

$$\begin{aligned} d_1 &= \frac{x + y + z}{3}, \\ d_2 &= \frac{xy + yz + zx}{3}, \\ d_3 &= xyz. \end{aligned} \quad (3)$$

With this notion, we formulate the first inequality, which establishes a handy relationship between the weighted elementary symmetric polynomials. It can be viewed as a refinement of the AM-GM inequality, which states $d_1 \geq d_n^{1/n}$.

2.1 MacLaurin's inequality

For all $x_1, \dots, x_n \geq 0$, we have

$$d_1 \geq d_2^{1/2} \geq \dots \geq d_n^{1/n}, \quad (4)$$

with equality at the k^{th} chain segment ($d_k^{1/k} \geq d_{k+1}^{1/(k+1)}$) holding if and only if $x_1 = x_2 = \dots = x_n$ or $n - k + 1$ variables from $\{x_1, \dots, x_n\}$ are zero.

MacLaurin's inequality will prove useful in our analysis, as it allows us to easily switch between different symmetric polynomials (by means of an estimate) occurring in the formula of the free energy density ε . These estimates will not be too rough, since we always have the possibility of equality when the variables are equal. Now, we come to the other inequality needed in this article.

2.2 Schur's inequality

For all $x, y, z \geq 0, \alpha > 0$, we have

$$x^\alpha(x-y)(x-z) + y^\alpha(y-z)(y-x) + z^\alpha(z-x)(z-y) \geq 0. \quad (5)$$

Equality holds if and only if two of $\{x, y, z\}$ are equal and the other is 0, or $x = y = z$.

For Schur's inequality consisting of a symmetric polynomial (if α is a natural number), it can be expressed as a polynomial in the (weighted) elementary symmetric polynomials. For $\alpha = 1$, one obtains the equivalent formulation [20]

$$d_3 + 3d_1^3 - 4d_1d_2 \geq 0, \quad (6)$$

again with the same equality conditions.

Notably, only cubic high symmetry directions can yield equality in Schur's and MacLaurin's inequalities. Hence, as shown in the following chapter, in our analysis it is not the symmetry of ε itself, but the applicability of these symmetric inequalities (to obtain equality in $\varepsilon \geq c$ or $\varepsilon \leq c$ for some c) which yields the easy/hard magnetic axes to coincide with the cubic high symmetry directions.

3 Results

In the following, "if and only if" is abbreviated by "iff." We incorporate the constraint $\sum_i \alpha_i^2 = 1$ into the free energy equation (Eq. (1)) by substituting $\alpha_1^2 = x/(x+y+z)$ (analogously for α_2, α_3), with $x, y, z \geq 0$ (and $x = y = z = 0$ being excluded, as this corresponds to zero magnetisation). Thus, we find the equivalent formulation

$$\varepsilon = \frac{K_1}{3} \frac{d_2}{d_1^2} + \frac{K_2}{27} \frac{d_3}{d_1^3}. \quad (7)$$

Now importantly, all of our following argumentations will be based on a simple reasoning: We can obtain the hard axes by finding some best constant c , for which $\varepsilon \leq c$ always holds (for $x, y, z \geq 0$) and by determining for which x, y, z equality is established. Analogously, the easy axes are found by determining the equality conditions for some best inequality $\varepsilon \geq c$. First, we consider the inequality for hard axes, i.e. $\varepsilon \leq c$, and multiply both sides by the positive $27d_1^3$, which yields the equivalent

$$9K_1d_1d_2 + K_2d_3 \leq 27cd_1^3. \quad (8)$$

If $K_1, K_2 > 0$, we can obtain this inequality for $c = K_2/27 + K_1/3$, since then

$$27cd_1^3 = \underbrace{K_2d_1^3}_{\geq K_2d_3} + \underbrace{9K_1d_1^3}_{\geq 9K_1d_1d_2} \geq K_2d_3 + 9K_1d_1d_2,$$

where for the first summand we used $K_2 \geq 0$ and $d_1^3 \geq d_3$ and for the second summand $K_1 \geq 0$ and $d_1^2 \geq d_2$. Thus, $\varepsilon \leq K_2/27 + K_1/3$, and since we only applied MacLaurin's inequality, this tells us that equality is obtained iff $x = y = z$. Hence, the hard axes are exactly along the directions $\langle 111 \rangle$. The easy axes we find by noting that since $K_1, K_2 > 0$ and $d_1, d_2, d_3 \geq 0$, we have $\varepsilon \geq 0$ with equality iff both summands in $\varepsilon = \frac{K_1}{3} \frac{d_2}{d_1^2} + \frac{K_2}{27} \frac{d_3}{d_1^3}$ (Eq. (7)) are 0, i.e. iff two of $\{x, y, z\}$ are 0. Hence, the easy axes are exactly along the $\langle 100 \rangle$ -directions.

$$\begin{aligned}
\varepsilon \geq c &\iff K_2 d_3 + 27|c|d_1^3 - 9|K_1|d_1d_2 \geq 0 & | : (9|K_1|/4) \\
&\iff t d_3 + r d_1^3 - 4d_1d_2 \geq 0, & \text{with } t := \frac{K_2}{9|K_1|/4}, \quad r := \frac{27|c|}{9|K_1|/4} \\
&\iff (t-1)d_3 + (r-3)d_1^3 + \underbrace{d_3 + 3d_1^3 - 4d_1d_2}_{\geq 0 \text{ by Schur (Eq. (6))}} \geq 0.
\end{aligned}$$

If $K_1, K_2 < 0$, we have the reverse inequalities (after multiplication by (-1) , hence interchanging $\varepsilon \leq c$ with $\varepsilon \geq c$), and thus the easy and hard axes are interchanged as compared to the case $K_1, K_2 > 0$.

Now, let $K_1 < 0, K_2 > 0$. We have for the hard axes by equation (8)

$$\varepsilon \leq c \iff K_2 d_3 \leq 27c \underbrace{d_1^3}_{\geq d_3} + 9|K_1| \underbrace{d_1}_{\geq d_3^{1/3}} \underbrace{d_2}_{\geq d_3^{2/3}}.$$

For $9|K_1| < K_2$, we can choose $c = (K_2 - 9|K_1|)/27$ to fulfil the inequality, with equality iff $x = y = z$. Hence, the hard axes correspond to $\langle 111 \rangle$. For $9|K_1| \geq K_2$, we need not choose a positive c anymore, particularly $\varepsilon \leq 0$ holds. Equality $\varepsilon = 0$ can indeed be obtained, exactly for the $\langle 100 \rangle$ -directions for $9|K_1| > K_2$ and exactly for the directions $\langle 100 \rangle, \langle 111 \rangle$ if $9|K_1| = K_2$.

For the easy axes, we consider

$$\varepsilon \geq c \iff 9K_1 d_1 d_2 + K_2 d_3 \geq 27c d_1^3.$$

Here, $c > 0$ is not possible, since for $x = y = 0, z \neq 0$, the left-hand-side is 0, but the right-hand-side would be greater than 0. So we assume there is $c \leq 0$. Then

See this equation at the top of this page

Observe that in the last step we did nothing else but bringing the inequality into a form where we can directly apply Schur's inequality (Eq. (6)). Now, we only consider the left two summands of the last line and distinguish different cases for t , yielding different easy axes:

For $t \geq 1$, the highest (non-positive) c (and thus smallest r) that one can choose to fulfil the inequality for all $x, y, z \geq 0$, corresponds to $r = 3$. Indeed, the choice $x = 0, y = z (\neq 0)$ renders the "Schur part" equal to 0 and hence the left-hand-side of the inequality equal to $(r-3)d_1^3$. If $r < 3$, this would be negative, in contradiction to the inequality " ≥ 0 ". Hence, for $r = 3$ respectively $c = -|K_1|/4$ we have $\varepsilon \geq c$ with equality iff $d_3 = 0$ and $d_3 + 3d_1^3 - 4d_1d_2 = 0$, hence iff two of $\{x, y, z\}$ are equal and the other 0. Thus, the easy axes are $\langle 110 \rangle$.

For $t < 1$, we have $(t-1)d_3 \leq 0$, thus necessarily $r - 3 \geq -(t-1)$. Else, if " $<$ " holds, then for $x = y = z (\neq 0)$, yielding " $= 0$ " in Schur's inequality and $d_1^3 = d_3$, the left-hand-side becomes negative, which is a contradiction. Furthermore, $r - 3 = -(t-1)$ works thanks to $d_1^3 \geq d_3$, thus $c = -(9|K_1| - K_2)/27$ and $\varepsilon \geq c$ with equality iff $x = y = z$. Hence, the easy axes are $\langle 111 \rangle$.

For $t = 1$, the smallest r (and thus smallest $|c|$) we can choose is $r = 3$ (for $r < 3$ and $x = 0, y = z (\neq 0)$, the left-hand-side is negative). Thus, the equality conditions from Schur's inequality yield both $\langle 110 \rangle$ and $\langle 111 \rangle$ as all the easy axes.

Thus, the case $K_1 < 0, K_2 > 0$ is finished. The reverse case $K_1 > 0, K_2 < 0$ yields the reverse easy/hard axes.

The only remaining cases are when one of the anisotropy constants is equal to 0. For $K_1 = 0$, we have

$$\varepsilon = \frac{K_2 d_3}{27 d_1^3} \begin{cases} \leq \frac{K_2}{27} & \text{for } K_2 > 0, \\ \geq \frac{K_2}{27} & \text{for } K_2 < 0, \end{cases}$$

with equality iff $x = y = z$ (from MacLaurin's inequality), corresponding to $\langle 111 \rangle$ directions. Furthermore,

$$\varepsilon \begin{cases} \geq 0 & \text{for } K_2 > 0, \\ \leq 0 & \text{for } K_2 < 0, \end{cases}$$

with equality iff $x = 0$ or $y = 0$ or $z = 0$. This corresponds to easy respectively hard planes $\{100\}$ for $K_2 > 0$ and $K_2 < 0$.

Finally, for $K_2 = 0$ we have

$$\varepsilon = \frac{K_1 d_2}{3 d_1^2},$$

yielding the analogous results as above for $K_1 = 0$ via MacLaurin's inequality. It remains to mention that the case $K_1 = K_2 = 0$ corresponds to isotropic behaviour.

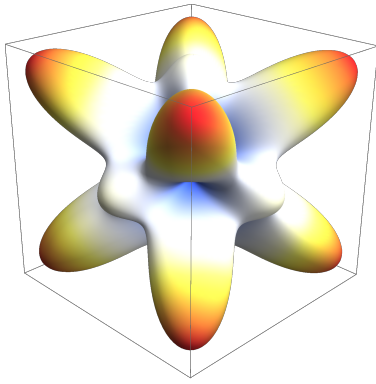
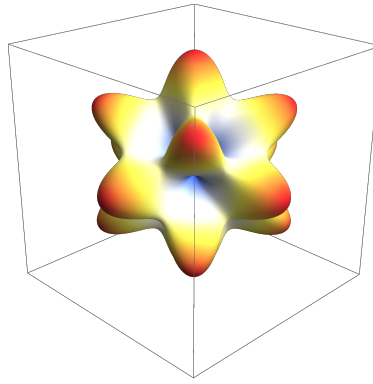
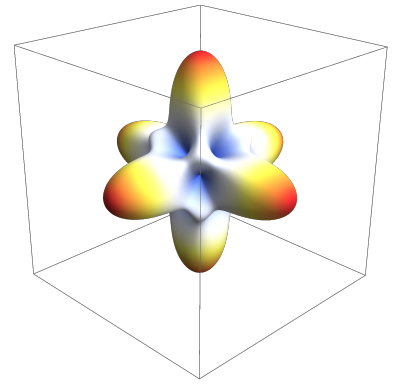
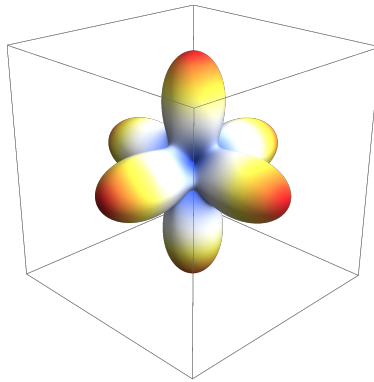
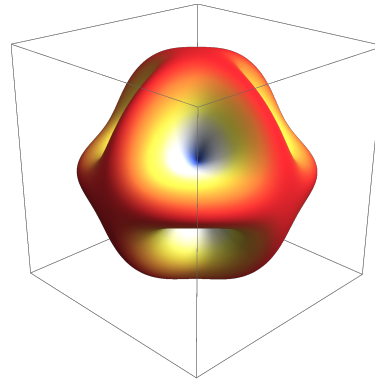
4 Discussion

All in all, we obtain the easy and hard axes for all possible values of K_1, K_2 . Note that half of the parameter space, namely the case of both K_1, K_2 having the same sign, is quite easy to handle, with only MacLaurin's inequality to be applied. The case of mixed signs for K_1, K_2 is more tedious, where multiple case distinctions need to be made. These distinctions are not artificial, though, as they yield different sets of easy and hard axes. The results are summarised in Table 1. Since the equality cases $K_2 = -9K_1$ and $K_2 = -9K_1/4$ are just transition cases, where two sets of easy or hard axes occur (see also Fig. 2b), we exclude them from the table for better clarity.

A prominent example for $K_1 < 0, K_2 < -9K_1/4$, yielding $\langle 111 \rangle$ as easy and $\langle 100 \rangle$ as hard axes, is fcc Ni, whereas bcc Fe usually corresponds to the opposite case [1]. The easy and hard axes can be visualised by a spherical plot,

Table 1. Easy and hard axes for all combinations of K_1, K_2 (except for the equality cases).

		$K_2 > -9K_1$	$-9K_1 > K_2 > -\frac{9K_1}{4}$	$K_2 < -\frac{9K_1}{4}$
$\underline{K_1 < 0}$	easy axes	$\langle 110 \rangle$	$\langle 110 \rangle$	$\langle 111 \rangle$
	hard axes	$\langle 111 \rangle$	$\langle 100 \rangle$	$\langle 100 \rangle$
		see Figure 2a	Figure 2c	Figure 2d
		$K_2 > 0$	$K_2 < 0$	
$\underline{K_1 = 0}$	easy axes/planes	$\{100\}$	$\langle 111 \rangle$	
	hard axes/planes	$\langle 111 \rangle$	$\{100\}$	
			see Figure 2e	
		$K_2 > -\frac{9K_1}{4}$	$-\frac{9K_1}{4} > K_2 > -9K_1$	$K_2 < -9K_1$
$\underline{K_1 > 0}$	easy axes	$\langle 100 \rangle$	$\langle 100 \rangle$	$\langle 111 \rangle$
	hard axes	$\langle 111 \rangle$	$\langle 110 \rangle$	$\langle 110 \rangle$


 (a) $K_2 = 15$

 (b) $K_2 = 9$

 (c) $K_2 = 5$

 (d) $K_2 = 0$

 (e) $K_1 = 0, K_2 = -1$
Fig. 2. Colour coded spherical plot (created with the software Wolfram Mathematica) of the magnitude of the shifted free energy density ε . The (here for simplicity unitless) value K_1 was always equal to -1 , except for the bottom right image (e), where $K_1 = 0$ was taken.

where the magnitude of ε is represented by the distance from the centre of the cube or by a colour code [1,10], see Figure 2 for a representation. For the magnitude to be meaningful, ε is shifted to values greater than 0: When a best inequality $c_2 \leq \varepsilon \leq c_1$ with $c_2 < 0$ holds, then $|c_2|$ is added to ε . Additionally, for clearer illustration, $1/5 \cdot (c_1 - c_2)$ is added, so that zero magnitude does not occur.

5 Dead end

First, we note that our analysis (unlike a calculus based analysis [3]) does not yield saddle points, nor local extrema that aren't global extrema. Furthermore, when including terms of higher order than six in ε , an akin full analysis solely based on these symmetric polynomial inequalities is not possible, given that the easy and hard axes do not always coincide with cubic high symmetry directions, as demonstrated in Figure 1b. However, they can be employed to find parameter sets where this coincidence occurs. For this, it might be necessary to use Schur's inequality (5) with higher exponents α .

For a complete analysis of the most relevant case (order eight, see Ref. [4]), other methods can be applied. One is the half-degree-principle [24], which also yields a solution for the case of sixth order, see Cor. 5.3 in reference [24]. It implies that the global minima of a symmetric polynomial with n non-negative variables and degree d under symmetric polynomial constraints $F_1, \dots, F_m$ are attained at points with at most $\max\{2, \lfloor \frac{d}{2} \rfloor, \deg F_1, \dots, \deg F_m\}$ distinct non-zero coordinates [25]. For $d = 4$ and degree 1 of the constraint (again substituting the squared direction cosines with non-negative variables), this number is 2, so that only the directions [100], [111], [11 $\bar{x}$] need to be evaluated. For [11 $\bar{x}$], the components of the normalised magnetisation vector are $(a, a, 1 - 2a)$ for $0 \leq a \leq 1/2$, and we have

$$\varepsilon = K_1 (a^2 + 2a(1 - 2a)) + K_2 a^2(1 - 2a) + K_3 (a^2 + 2a(1 - 2a))^2. \quad (9)$$

Finding the minima of this expression (possibly after normalising K_3 to 1) and comparing them with ε for the [100], [111] directions, i.e. 0 respectively $K_1/3 + K_2/27 + K_3/9$, is straightforward but tedious.

6 Conclusion

We demonstrate an efficient derivation of the magnetic easy and hard axes in cubic magnetic crystals, by analysing the symmetric anisotropy energy density ε (up to sixth order) just using basic mathematical inequalities that are well-suited to the structure of the problem. Particularly, we apply symmetric polynomial inequalities, for which equality is obtained only for cubic high symmetry directions. The very applicability of these inequalities to yield $\varepsilon \geq c$ or $\varepsilon \leq c$ for some best constant c , and determining the equality conditions, then establishes the easy

and hard axes to also coincide with these high symmetry directions.

Appendix A: Proof of the inequalities

Here we present mathematical proofs of the inequalities used in this manuscript, partly following reference [16].

A.1 Proof of MacLaurin's inequality

There is an interesting proof [26] using induction on the number of variables in the d_k . For this, we first note that the s_k are the coefficients of t^{n-k} of the polynomial $P(t) = \prod_{k=1}^n (t + x_k)$ (Vieta's theorem). Now, we reduce the number of variables using a trick, namely, we consider the derivative P' , which can be written as $P'(t) = n \prod_{k=1}^{n-1} (t + \tilde{x}_k)$. Just as the x_k , also the $\tilde{x}_k$ are non-negative, since a zero of P' lies between two distinct consecutive zeros of P (by Rolle's theorem) or coincides for multiple zeros. Furthermore, the symmetric polynomials $\tilde{s}_k$ in $\tilde{x}_1, \dots, \tilde{x}_{n-1}$ are the coefficients of t^{n-k-1} in $P'(t)/n$ and therefore (by comparison with the coefficients of P'/n evaluated from P) equal to $(n - k)/n \cdot s_k$. Thus, $\tilde{d}_k = \frac{1}{\binom{n-1}{k}} \frac{n-k}{n} s_k = d_k$. This is a remarkable property, because we can now conclude that an inequality in d_k is true if it is true for $\tilde{d}_k$, which contains one variable less than d_k . Thus, with the induction start $(x_1 + x_2)/2 \geq \sqrt{x_1 x_2}$ (which is true by the AM-GM inequality) and by showing the last inequality in the chain for general n , namely $d_{n-1}^{1/(n-1)} \geq d_n^{1/n}$, we can "work our way up" and the induction proof is finished. Both sides of the last inequality are homogeneous of degree 1, so that every x_k can be multiplied by some fixed positive number without changing the inequality. Hence, if none of the x_k is 0 (else the inequality would trivially follow since $d_n = 0$), without loss of generality it can be assumed $\prod_k x_k = 1$. Thus, the inequality is equivalent to $1/n \cdot \sum_k 1/x_k \geq 1$, which is true by the AM-GM inequality. Similarly to the inequalities, also the equality cases follow from the induction. The condition $x_1 = \dots = x_n$ follows from the equality condition in the AM-GM inequality, and the other one comes from the fact that d_k contains sums of products of k variables, where all summands are zero exactly when at least $n - k + 1$ variables are 0 (pigeonhole principle). Then, from the representation of P and P' with the x_k respectively $\tilde{x}_k$, and using that a multiple zero in P (with multiplicity $r > 1$) will be a multiple zero of P' (with multiplicity $r - 1$), the equality cases are obtained.

A.2 Proof of Schur's inequality

Thanks to the symmetry in x, y, z , we can without loss of generality assume some order, $x \leq y \leq z$. Then, the first summand in Schur's inequality is obviously non-negative. For the other two summands, we can factor out $(z - y)$, yielding $(z - y)(z^\alpha(z - x) - y^\alpha(y - x))$, which is non-negative due to $z \geq y$ and $z - x \geq y - x$. The equality condition can be easily checked.

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