\(L_\infty\)-algebras and their cohomology

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Abstract. We give an overview of different characterisations of \(L_\infty\)-structures in terms of symmetric brackets and (co)differentials on the symmetric (co)algebra. We then do the same for their representations (up to homotopy) and approach \(L_\infty\)-algebra cohomology using the commutator bracket on the space of coderivations of the symmetric coalgebra. This leads to abelian extensions of \(L_\infty\)-algebras by 2-cocycles.

Keywords: \(L_\infty\)-algebras / representations (up to homotopy) / \(L_\infty\)-algebra cohomology / abelian extensions by 2-cocycles

1 Introduction

\(L_\infty\)-algebras (also called strongly homotopy Lie algebras) were first introduced in [1] and [2] and are a generalisation of graded Lie algebras in which a system of antisymmetric \(n\)-ary brackets satisfies a generalised Jacobi identity. The first part of this article serves as a self-contained introduction to \(L_\infty\)-algebras, in which we discuss different characterisations of \(L_\infty\)-algebras and their representations (up to homotopy), closely following [3].

The \(L_\infty\)-algebra cohomology with values in the adjoint representation was introduced in [4] using a Lie bracket on the space of cochains. We extend this approach to arbitrary representations, which leads to a characterisation of certain \(L_\infty\)-algebras as abelian extensions of \(L_\infty\)-algebras by 2-cocycles. This generalises a theorem from [5] that characterises certain \(L_\infty\)-algebras in terms of Lie algebra cohomology.

This article is largely based on my same-titled Bachelor’s thesis, which I wrote under the supervision of Chenchang Zhu at the University of Göttingen in 2018.

2 Mathematical background

In this section, we discuss exterior and symmetric powers, algebras and coalgebras in the graded framework. In particular, we show that antisymmetric and symmetric maps are related by a shift in degree and that coderivations of the symmetric coalgebra are in one-to-one correspondence with their weight one components. These results are later key to the characterisations of \(L_\infty\)-structures in terms of symmetric brackets and codifferentials. The main references for this section are [3,4,6,7].

2.1 Graded vector spaces

A graded vector space is a vector space \(V\) together with a decomposition \(V \cong \bigoplus_{p \in \mathbb{Z}} V_p\) for a family of vector spaces \(\{V_p\}_{p \in \mathbb{Z}}\). An element \(v \in V_p\) is then called homogeneous of degree \(p\) and we write \(|v| = p\).

Here and subsequently, we assume all vector spaces to be over a fixed ground field \(k\) of characteristic zero. We always denote by \(V\) and \(W\) graded vector spaces and by \(v_1, \ldots, v_n \in V\) arbitrary homogeneous elements.

A linear map \(f : V \to W\) is called homogeneous (of degree \(p\)) if there is \(p \in \mathbb{Z}\) such that \(f(V_n) \subset W_{n+p}\) for all \(n \in \mathbb{Z}\). We denote by \(\text{Hom}_p(V,W)\) the vector space of all homogeneous linear maps \(V \to W\) of degree \(p\) and by \(\text{Hom}(V,W)\) the graded vector space \(\bigoplus_{p \in \mathbb{Z}} \text{Hom}_p(V,W)\). Elements in \(\text{Hom}_0(V,W)\) are also called degree preserving.

Note that we can identify ungraded vector spaces with graded ones that are concentrated in degree zero, that is \(V_k = 0\) for \(k \neq 0\).

There is a canonical grading on the direct sum of \(V\) and \(W\) given by \((V \oplus W)_p = V_p \oplus W_p\). The isomorphism

\[V \otimes W \cong \bigoplus_{p \in \mathbb{Z}} \bigoplus_{i+j=p} (V_i \otimes W_j)\]

allows us to define a grading on \(V \otimes W\) by

\[(V \otimes W)_p = \bigoplus_{i+j=p} (V_i \otimes W_j)\]
This extends to a grading on $V^\otimes n := \bigotimes_{i=1}^n V$ given by

$$(V^\otimes n)_p = \bigoplus_{i_1 + \ldots + i_n = p} V_{i_1} \otimes \ldots \otimes V_{i_n}.$$  

We denote by $\tau_{V,W}$ the linear degree preserving map

$$\tau_{V,W} : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v.$$  

If $f \in \text{Hom}(V,W)$ and $g \in \text{Hom}(V',W')$ are homogeneous for graded vector spaces $V'$ and $W'$, we define the linear map $f \otimes g : V \otimes V' \to W \otimes W'$ by

$$(f \otimes g)(v \otimes v') = (-1)^{|v||g|} f(v) \otimes g(v')$$  \hspace{1cm} (1)$$

for $v \in V$, $v' \in V'$ homogeneous. Note that $|f \otimes g| = |f| + |g|$. This generalises to tensor products of three or more vector spaces in the obvious way and we abbreviate $f \otimes \ldots \otimes f$.

For the composition of such functions, (1) implies

$$(f' \otimes g') \circ (f \otimes g) = (-1)^{|g'||f|}(f' \circ f) \otimes (g' \circ g),$$  \hspace{1cm} (2)$$

when $f'$ and $g'$ are homogeneous linear maps with domains $W$ and $W'$, respectively.

When working in the framework of graded vector spaces, the general rule of thumb for the signs is that whenever two “graded symbols” of degree $p$ and $q$, respectively, change their order in an equation, there should be the sign $(-1)^{|p||q|}$. This is called the Koszul sign convention.

We denote by $\mathfrak{S}_n$, the symmetric group, the group of all permutations of the set $\{1, \ldots, n\}$, and by $s_i \in \mathfrak{S}_n$ for $1 \leq i \leq n-1$ the transposition with $s_i(i) = i + 1$ and $s_i(i + 1) = i$. There are two canonical right actions of $\mathfrak{S}_n$ on $V^\otimes n$. These are given on the generating subset $\{s_1, \ldots, s_{n-1}\} \subset \mathfrak{S}_n$ by

$$\xi(s_i)(v_1 \otimes \ldots \otimes v_n) = (-1)^{|v_i||v_{i+1}|} v_1 \otimes \ldots \otimes v_{i-1}$$

$$\otimes v_{i+1} \otimes v_i \otimes v_{i+2} \otimes \ldots \otimes v_n, \quad \chi(s_i)(v_1 \otimes \ldots \otimes v_n) = (-1)^{|v_i||v_{i+1}|} v_1 \otimes \ldots \otimes v_{i-1}$$

$$\otimes v_{i+1} \otimes v_i \otimes v_{i+2} \otimes \ldots \otimes v_n.$$  

We call $\xi$ and $\chi$ the (graded) symmetric and (graded) antisymmetric action of $\mathfrak{S}_n$ on $V^\otimes n$, respectively. Note that $\xi(\sigma)$ is degree preserving for all $\sigma \in \mathfrak{S}_n$ as

$$\xi(\sigma)(v_1 \otimes \ldots \otimes v_n) = \pm v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}.$$  \hspace{1cm} (3)$$

We then denote the sign in (3) by $\varepsilon(\sigma; v_1, \ldots, v_n)$ and similarly by $\chi(\sigma; v_1, \ldots, v_n)$ the sign such that

$$\chi(\sigma)(v_1 \otimes \ldots \otimes v_n) = \chi(\sigma; v_1, \ldots, v_n)v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}.$$  

We abbreviate $\varepsilon(\sigma; v_1, \ldots, v_n)$ and $\chi(\sigma; v_1, \ldots, v_n)$ to $\varepsilon(\sigma)$ and $\chi(\sigma)$, when no confusion arise.

Let $U_S \subset V^\otimes n$ be the graded subspace spanned by all elements of the form

$$v_1 \otimes \ldots \otimes v_n - \xi(\sigma)(v_1 \otimes \ldots \otimes v_n)$$

for $\sigma \in \mathfrak{S}_n$. The space $S^n(V) := V^\otimes n/U_S$ is called the $n$th symmetric power of $V$. Similarly, the $n$th exterior power of $V$ is defined as the quotient of $V^\otimes n$ by the graded subspace spanned by all elements of the form

$$v_1 \otimes \ldots \otimes v_n - \chi(\sigma)(v_1 \otimes \ldots \otimes v_n)$$

for $\sigma \in \mathfrak{S}_n$ and is denoted by $\wedge^n V$.

An $n$-linear map $f : V^n \to W$ is called (graded) symmetric if for all $\sigma \in \mathfrak{S}_n$

$$f(v_1, \ldots, v_n) = \varepsilon(\sigma)f(v_{\sigma(1)}, \ldots, v_{\sigma(n)})$$

holds. We can write this conveniently as $f \circ \xi(\sigma) = f$. Similarly, $f$ is called (graded) antisymmetric if $f \circ \chi(\sigma) = f$ for all $\sigma \in \mathfrak{S}_n$.

**Proposition 1.** Let $f : V^\otimes n \to W$ be a symmetric linear map. There is a unique linear map $\varphi : S^n(V) \to W$ such that the following diagram commutes:

$$\begin{diagram}
V^\otimes n & \xrightarrow{\pi_S} & S^n(V) \\
\pi_S & \xrightarrow{f} & \xrightarrow{\varphi} W
\end{diagram}$$

where $\pi_S : V^\otimes n \to S^n(V)$ is the canonical projection.

**Proof.** As $f$ is symmetric, it vanishes on the generators of $U_S$ and factors through $\pi_S$ to a linear map $\varphi : S^n(V) \to W$ such that the diagram above commutes. This map is unique as $\pi_S$ is surjective.

**Remark 2.** As the symmetric $\mathfrak{S}_n$-action on $V^\otimes n$ is degree preserving, $S^n(V)$ inherits a canonical grading from $V^\otimes n$ such that $\pi_S$ is degree preserving. It is then immediate that if $f$ is homogeneous in Proposition 1, so is the map $\varphi$ and $|\varphi| = |f|$. As $\pi_S$ is symmetric by construction of $S^n(V)$, Proposition 1 yields an isomorphism between the subspace of $\text{Hom}(V^\otimes n, W)$ consisting of all symmetric maps and $\text{Hom}(S^n(V), W)$. An analogue of Proposition 1 holds for $\wedge^n V$ and induces an isomorphism between the subspace $\text{Hom}(V^\otimes n, W)$ of all anti-symmetric maps and $\text{Hom}(\wedge^n V, W)$.

An element in $V^\otimes n$ is called symmetric if it is invariant under the symmetric $\mathfrak{S}_n$-action on $V^\otimes n$. We claim that $S^n(V)$ is isomorphic to the subspace of $V^\otimes n$ of all symmetric elements. Indeed, letting $v_1 \vee \ldots \vee v_n$ denote the image of $v_1 \otimes \ldots \otimes v_n$ under $\pi_S$, the linear map

$$\varphi : S^n(V) \to \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}$$

is well-defined and satisfies $\pi_S \circ \varphi = \text{id}_{S^n(V)}$ and $\varphi \circ \pi_S = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \xi(\sigma)$. As the latter is a projection of $V^\otimes n$ onto said subspace, the claim follows. A similar statement clearly holds for $\wedge^n V$.

For $n \in \mathbb{Z}$, we define the graded vector space $V[n]$ to be the vector space $V$ with the grading defined by $V[n]_p = V_{p+n}$. We denote by $\downarrow^n : V \to V[n]$ the identity map on $V$, which becomes a linear isomorphism of degree $-n$, and by $\uparrow^n$ its inverse. We abbreviate $\downarrow^1$ and $\uparrow^1$.
to $\downarrow$ and $\uparrow$, respectively. Note that $(\downarrow^{\otimes k})^{-1} = (-1)^{\frac{k(k-1)}{2}} \downarrow^{\otimes k}$ as a consequence of (2).

**Proposition 3** (The décalage isomorphism). For $\sigma \in \mathfrak{S}_n$, $\varepsilon(\sigma) \downarrow^{\otimes n} = \downarrow^{\otimes n} \circ \varepsilon(\sigma)$.  

There is then a degree preserving isomorphism

$$S^n(V[1]) \cong \left( \bigwedge^n V \right) [n].$$  

**Proof.** Note that for the first part, we only have to check (4) on the generating subset $\{s_1, \ldots, s_{n-1}\} \subset \mathfrak{S}_n$. This is an easy computation left to the reader. Let $\pi_A: V^{\otimes n} \to \bigwedge^n V$ be the canonical projection. The linear maps

$$\pi_S \circ \downarrow^{\otimes n}: V^{\otimes n} \to S(V[1]),$$

$$(1) \quad (-1)^{\frac{n(n-1)}{2}} \pi_A \circ \downarrow^{\otimes n}: (V[1])^{\otimes n} \to \bigwedge^n V$$

are then antisymmetric and symmetric, respectively. The induced linear maps between $S(V[1])$ and $\bigwedge^n V$ are then easily seen to be inverse to each other. As these maps are of degree $-n$ and $n$, respectively, we obtain a degree preserving isomorphism $S(V[1]) \cong (\bigwedge^n V) [n]$.

**Corollary 4.** There is for each $p \in \mathbb{Z}$ a one-to-one correspondence between symmetric linear maps $\lambda: (V[1])^{\otimes n} \to S(V[1])$ of degree $p$ and antisymmetric linear maps $l: V^{\otimes n} \to V$ of degree $p + 1 - n$ given by

$$l = \uparrow \circ \lambda \circ \downarrow^{\otimes n},$$

$$\lambda = (-1)^{\frac{n(n-1)}{2}} \downarrow \circ l \circ \uparrow^{\otimes n}.$$  

A **differential** on the graded vector space $V$ is a linear map $d: V \to V$ of degree one such that $d^2 = 0$. We then call the pair $(V, d)$ a **differential graded vector space** (DG vector space for short). A **homomorphism between DG vector spaces** $(V, d)$ and $(W, d')$ is a degree preserving linear map $f: V \to W$ such that $d' \circ f = f \circ d$.

DG vector spaces are sometimes called callchain complexes. Given a cochain complex $(V, d)$, one then calls an element $v \in V_n$ an **n-cocycle** if $d(v) = 0$ and an **n-cocohomology** if $v = d(w)$ for some $w \in V_{n-1}$. The graded vector space $H(V) = \ker(d)/\text{im}(d)$ measures the non-exactness of the sequence

$$\cdots \overset{d}{\to} V_{n-1} \overset{d}{\to} V_n \overset{d}{\to} V_{n+1} \overset{d}{\to} \cdots$$

and is called the **cohomology** of $(V, d)$. We then call $H_n(V) = \frac{n\text{-cocycles}}{n\text{-cohomologies}}$ the $n$th **cohomology group**.

### 2.2 Graded algebras

By an algebra we mean a vector space $A$ together with a linear map $\mu: A \otimes A \to A$; the multiplication $\mu$ is in general not assumed to be associative.

A **graded algebra** $A$ is an algebra that is also a graded vector space in which the multiplication is degree preserving. If also $ab = (-1)^{|a||b|}ba$ for all $a, b \in A$, we call $A$ a **(graded) commutative**. A homomorphism of graded algebras is a degree preserving algebra homomorphism. A (two-sided) ideal $I$ in $A$ is called **homogeneous** if $I \subset A$ is a graded subspace. Note that an ideal is homogeneous if and only if it is spanned by homogeneous elements.

**Remark 5.** If $I \subset A$ is a homogeneous ideal, the canonical isomorphism $A/I \cong \bigoplus_{n \in \mathbb{Z}} A_n/I_n$ makes $A/I$ into a graded algebra such that the canonical projection $A \to A/I$ is a homomorphism of graded algebras.

**Remark 6.** Let $A$ and $B$ be two graded associative algebras. The multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{|a||a'|}aa' \otimes bb'.$$

for $a, a' \in A$, $b, b' \in B$ homogeneous makes $A \otimes B$ into a graded associative algebra. If $A$ and $B$ are both unital/commutative, then so is $A \otimes B$.

**Example 7** (The tensor algebra). We denote by $T(V) : = \bigoplus_{n \geq 0} V^{\otimes n}$ the **tensor algebra** of $V$. It carries the multiplication induced by the canonical isomorphism $V^{\otimes n} \otimes V^{\otimes m} \cong V^{\otimes (n+m)}$, making it into a unital associative algebra. The grading on $T(V)$ induced by the grading on $V^{\otimes n}$ is given by

$$T(V)_p = \bigoplus_{i_1 + \cdots + i_n = p} V_{i_1} \otimes \cdots \otimes V_{i_n}$$

and is called the **interior grading**. On the other hand, $T(V)$ carries the grading given by $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$, which is called the **exterior grading** or **grading by weight**. If not specified otherwise, we always understand $T(V)$ to carry its interior grading. Note that both gradings make $T(V)$ into a graded algebra.

**Example 8** (The symmetric and exterior algebra). Let $I_S \subset T(V)$ be the two-sided homogeneous ideal generated by elements of the form $v_1 \otimes v_2 - (-1)^{|v_1||v_2|}v_2 \otimes v_1$. We call $S(V) : = T(V)/I_S$ the **symmetric algebra** of $V$. Similarly, the **exterior algebra** of $V$, denoted by $\bigwedge V$, is defined as the quotient of $T(V)$ by the two-sided homogeneous ideal generated by elements of the form $v_1 \otimes v_2 + (-1)^{|v_1||v_2|}v_2 \otimes v_1$. We denote the multiplication in $S(V)$ by $\bigwedge V$ by $\vee$ and $\wedge$, respectively. Note that $S(V)$ and $\bigwedge V$ also admit an **exterior grading or grading by weight**. As $V^{\otimes n} \cap I_S = U_S$, we have $S(V) = \bigoplus_{n \geq 0} S^n(V)$ and similarly $\bigwedge V = \bigoplus_{n \geq 0} \wedge^n V$.

It is easy to see that if $A$ is a graded unital associative algebra, there is for each linear degree preserving map $f: V \to A$ a unique homomorphism of unital graded algebras $\varphi: T(V) \to A$ that agrees on $V$ with $f$ (see for example [6], Proposition 1.1.1). It is then immediate that if $A$ is commutative, $\varphi$ factors to a unique homomorphism of unital graded algebras $S(V) \to A$. Applying this to the linear map

$$V \oplus W \to S(V) \otimes S(W), \quad (v, w) \mapsto v \otimes 1 + 1 \otimes w$$

yields a homomorphism of graded unital algebras $S(V \oplus W) \to S(V) \otimes S(W)$ that is easily seen to be an isomorphism with inverse $S(V) \otimes S(W) \to S(V \oplus W)$, $v \otimes w \mapsto v \vee w$. With a slight modification of the sign in
homomorphism of DG algebras vector space of all derivations ofExample 10.\[ab\] is an element this makes is a DGLA. In particular, for\[satisfies degree one element such thatExample 12.\[for short) is a DG algebra in which the underlying algebra\[define the derivation of the graded algebra\[satisfying the bracket\[A\] 2.3 Graded Lie algebras and unshuffle permutations

Definition 9. A graded Lie algebra is a graded vector space \(L\) together with a (graded) antisymmetric degree preserving linear map \([\cdot,\cdot]\): \(L \otimes L \to L\) called the Lie bracket satisfying the (graded) Jacobi identity\[for all \(x, y, z \in L\) homogeneous.

If \(L\) is ungraded, we recover the usual definition of a Lie algebra. Note that (9) is nothing else than \([x, \cdot]\) being a derivation of the graded algebra \((L, [\cdot,\cdot])\).

Example 10. For a graded associative algebra \(A\), we define the graded commutator \([\cdot,\cdot]: A \otimes A \to A\) by \([a, b] = ab - (-1)^{|a||b|}ba\) for \(a, b \in A\) homogeneous. This makes \(A\) into a graded Lie algebra. In particular, \(\mathfrak{gl}(V) := \text{Hom}(V, V)\) becomes a graded Lie algebra. If \(V\) is itself a graded algebra, one can check that \(\text{Der}(V) \subset \mathfrak{gl}(V)\) is a Lie subalgebra.

Definition 11. A differential graded Lie algebra (DGLA) for short) is a DG algebra in which the underlying algebra is a graded Lie algebra.

Example 12. Let \(L\) be a graded Lie algebra and \(x \in L\) a degree one element such that \(\frac{1}{2}[x, x] = 0\). Then \(d := [x, \cdot]\) satisfies \(d^2 = 0\) by the Jacobi identity (9) and \((L, [\cdot,\cdot], d)\) is a DGLA. In particular, for \((V, \partial)\) a DG vector space, this makes \((\mathfrak{gl}(V), [\cdot,\cdot], [\partial, \cdot])\) canonically into a DGLA as \(\frac{1}{2} [\partial, \partial] = d^2 = 0\).

Definition 13. For a DGLA \((L, [\cdot,\cdot], d)\), a Maurer–Cartan element is an element \(x \in L\) of degree one such that \[d(x) + \frac{1}{2} [x, x] = 0.\] The equation (10) is called the Maurer–Cartan equation.

Example 14. Let \((L, [\cdot,\cdot], d = [x, \cdot])\) be as in Example 12. For \(y \in L\) of degree one, we then have \(\frac{1}{2}[x+y, x+y] = 0\) if and only if \(y\) satisfies the Maurer–Cartan equation.

For \(0 \leq i \leq n\), an \((i, n - i)\)-unshuffle is a permutation \(\sigma \in \mathfrak{S}_n\) satisfying \(\sigma(1) < \cdots < \sigma(i)\) and \(\sigma(i + 1) < \cdots < \sigma(n)\). Following the notation in [3], we denote the set of all \((i, n - i)\)-unshuffles by \(\mathfrak{S}_{i, n-i} \subset \mathfrak{S}_n\). Using the antisymmetry of the Lie bracket, one can rewrite (9) as
\[
\sum_{\sigma \in \mathfrak{Sh}_{i, n-i}} \chi(\sigma) [x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)} = 0
\] for all \(x_1, x_2, x_3 \in L\) homogeneous.

Lemma 15. Each element \(\sigma \in \mathfrak{S}_n\) has for each \(i \in \{0, \ldots, n\}\) a unique decomposition \(\sigma = \tau(\alpha, \beta)\), where \(\tau \in \mathfrak{Sh}_{n-i}\) and \((\alpha, \beta) \in \mathfrak{S}_i \times \mathfrak{S}_{n-i}\). Here, \(\mathfrak{S}_i \times \mathfrak{S}_{n-i}\) is considered as a subgroup of \(\mathfrak{S}_n\) in the obvious way.

Proof. Clearly, \(\tau\) has to be the unique \((i, n - i)\)-unshuffle such that \(\{\tau(1), \ldots, \tau(i)\} = \{\sigma(1), \ldots, \sigma(i)\}\) and \(\{\tau(i + 1), \ldots, \tau(n)\} = \{\sigma(i + 1), \ldots, \sigma(n)\}\). We then have \(\tau^{-1}\sigma \in \mathfrak{S}_i \times \mathfrak{S}_{n-i}\).

2.4 Graded coalgebras

A (graded) coalgebra \((C, \Delta)\) is a graded vector space \(C\) together with a degree preserving linear map \(\Delta: C \to C \otimes C\) called the coproduct. If the diagram
\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
\downarrow{\Delta} & & \downarrow{\Delta \otimes \text{id}_C} \\
C \otimes C & \xrightarrow{\text{id}_C \otimes \Delta} & C \otimes C \otimes C
\end{array}
\] commutes, \(C\) is called coassociative. We call \(C\) counital if there is a degree preserving linear map \(\varepsilon: C \to \mathbb{k}\) such that the diagram
\[
\begin{array}{ccc}
\mathbb{k} \otimes C & \xleftarrow{\varepsilon \otimes \text{id}_C} & C \otimes C \\
\downarrow{\Delta} & & \downarrow{\Delta \otimes \varepsilon} \\
C & \xrightarrow{\Delta} & C \otimes \mathbb{k}
\end{array}
\] commutes. The map \(\varepsilon\) is then called the counit of \(C\). If \(\tau_{C,C} \circ \Delta = \Delta\), then \(C\) is called cocommutative. A linear degree preserving map \(f: C \to D\) between coalgebras \((C, \Delta_C)\) and \((D, \Delta_D)\) is called a homomorphism of coalgebras if
\[
(f \otimes f) \Delta_C = \Delta_D \circ f.
\]
If \(C\) and \(D\) are counital with counits \(\varepsilon\) and \(\eta\), respectively, and if \(f\) also satisfies \(\eta \circ f = \varepsilon\), we call \(f\) a homomorphism of counital coalgebras.
For a coassociative coalgebra \((C, \Delta)\) and \(n \in \mathbb{N}\), we define the \textit{iterated coproduct} \(\Delta^n : C \rightarrow C \otimes (n+1)\) by \(\Delta^0 = \text{id}_C\) and \(\Delta^n = (\Delta \otimes \text{id}_C \otimes \ldots \otimes \text{id}_C)\Delta^{n-1}\) for \(n \geq 1\). It is convenient to then use \textit{Sweedler notation} and to write \(\Delta^n(x) \in C \otimes (n+1)\) for \(x \in C\) as

\[
\Delta^n(x) = \sum x^{(1)} \otimes \ldots \otimes x^{(n+1)}.
\]

In this notation, for example, the condition for \(C\) to be cocommutative becomes \(\sum x^{(1)} \otimes x^{(2)} = \sum (-1)^{|x^{(2)}|} x^{(2)} \otimes x^{(1)}\) for all \(x \in C\).

**Lemma 16.** Let \((C, \Delta_C)\) be a coassociative coalgebra. Then for all \(p, q \in \mathbb{N}\),

\[
(\Delta_C^p \otimes \Delta_C^q) \Delta_C = \Delta_C^{p+q+1}.
\]

If \((D, \Delta_D)\) is another coassociative coalgebra and if \(f : C \rightarrow D\) is a coalgebra homomorphism,

\[
f^{\otimes(n+1)} \circ \Delta_C^n = \Delta_D^n \circ f
\]

holds for all \(n \in \mathbb{N}\).

**Proof.** One obtains (15) and (16) by iterating (12) and (14); the details are left to the reader or can be found in (7), Lemma-Definition VIII.10).

### 2.4.2 Examples of coalgebras

There is a coproduct \(\Sigma_A\) on \(T(V) := \bigoplus_{n \geq 1} V^\otimes n\) given by

\[
\Sigma_A(v_1 \ldots v_n) = \sum_{i=1}^{n-1} (v_1 \ldots v_i) \otimes (v_{i+1} \ldots v_n),
\]

where we now denote the multiplication in \(T(V)\) by concatenation to avoid ambiguities. This makes \(T(V)\) into a coassociative graded coalgebra. The induced coaugmented coalgebra \((T(V), \Delta_A)\) is called the \textit{tensor coalgebra}. Inductively, one finds

\[
\Sigma_A^n(v_1 \ldots v_n) = \sum_{1 \leq i_1 < \ldots < i_m < n} (v_1 \ldots v_{i_1}) \otimes \ldots \otimes (v_{i_m+1} \ldots v_n),
\]

which shows that \(T(V)\) is conilpotent.

**Proposition 17.** Let \((C, \Delta)\) be a conilpotent coalgebra and \(f : C \rightarrow V\) a linear degree preserving map. There is a unique homomorphism of coaugmented coalgebras \(\tilde{f} : C \rightarrow T(V)\) such that \(f = \text{pr}_V \circ \tilde{f}\), where here and subsequently, \(\text{pr}_V\) denotes the projection onto a subspace under a given decomposition.

**Proof.** It clearly suffices to show that there is a unique homomorphism of coalgebras \(\tilde{f} : C \rightarrow T(V)\) satisfying \(\text{pr}_V \circ \tilde{f} = f\). For the uniqueness, assume that there is such \(\tilde{f}\). By Lemma 16, we have

\[
\tilde{f}^{\otimes(n+1)} \circ \Sigma_A^n = \Sigma_A^n \circ \tilde{f}
\]

for all \(n \in \mathbb{N}\). Composing both sides with \(\text{pr}_V^{\otimes(n+1)}\) and noting that \(\text{pr}_V^{\otimes(n+1)} \circ \Sigma_A^n = \text{pr}_V^{\otimes(n+1)} \circ \tilde{f}^{\otimes(n+1)} \circ \Sigma_A^n = f^{\otimes(n+1)} \circ \Sigma_A^n\), we obtain

\[
\text{pr}_V^{\otimes(n+1)} \circ \tilde{f} = \text{pr}_V^{\otimes(n+1)} \circ \tilde{f}^{\otimes(n+1)} \circ \Sigma_A^n = f^{\otimes(n+1)} \circ \Sigma_A^n.
\]
\[ \Delta_A(N(v_1 \ldots v_n)) = \sum_{i=1}^{n-1} \sum_{\sigma \in S_n} \varepsilon(\sigma)(v_{\sigma(1)} \ldots v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \ldots v_{\sigma(n)}) \]
\[ = \sum_{i=1}^{n-1} \sum_{\tau \in Sh_{n-i}} \sum_{(\alpha,\beta) \in \mathfrak{S}_i \times \mathfrak{S}_{n-i}} \varepsilon(\tau(\alpha,\beta))(v_{\tau(\alpha,\beta)(1)} \ldots v_{\tau(\alpha,\beta)(i)}) \otimes (v_{\tau(\alpha,\beta)(i+1)} \ldots v_{\tau(\alpha,\beta)(n)}) \]
\[ = \sum_{i=1}^{n-1} \sum_{\tau \in Sh_{n-i}} \varepsilon(\tau) \sum_{(\alpha,\beta) \in \mathfrak{S}_i \times \mathfrak{S}_{n-i}} \varepsilon(\alpha)(v_{\tau(1)} \ldots v_{\tau(i)}) \otimes \varepsilon(\beta)(v_{\tau(i+1)} \ldots v_{\tau(n)}) \]
\[ = \sum_{i=1}^{n-1} \sum_{\tau \in Sh_{n-i}} \varepsilon(\tau)N(v_1 \ldots v_i) \otimes N(v_{i+1} \ldots v_n). \]

This shows that \( \hat{f} \) is completely determined by \( f \) and therefore unique. For the existence, consider the linear map \( \sum_{n=0}^{\infty} \Delta^n : C \rightarrow T(C) \). This is well-defined as \( C \) is comultiplicative. A straightforward computation using Lemma 16 and \( \Delta_A|_V = 0 \) shows that

\[ \left( \sum_{n=0}^{\infty} \Delta_n \otimes \sum_{n=0}^{\infty} \Delta_n \right) \Delta = \Delta_A \circ \sum_{n=0}^{\infty} \Delta_n, \]

so that \( \sum_{n=0}^{\infty} \Delta^n \) is a coalgebra homomorphism. As \( T(f) = \bigoplus_{n \geq 1} f^{\otimes n} : T(C) \rightarrow \bar{T}(V) \) is easily seen to be also a coalgebra homomorphism, \( \hat{f} := \bar{T}(f) \circ \sum_{n=0}^{\infty} \Delta^n : C \rightarrow \bar{T}(V) \) is a homomorphism of coalgebras with \( \text{pr}_V \circ \hat{f} = f \).

Let \( S(V) := \bigoplus_{n \geq 1} S^n(V) \). Consider the linear maps
\[ \pi : T(V) \rightarrow \bar{S}(V), \quad v_1 \otimes \ldots \otimes v_n \mapsto \frac{1}{n!} v_1 \ldots v_n, \]
\[ N : \bar{S}(V) \rightarrow \bar{T}(V), \]
\[ v_1 \ldots v_n \mapsto \sum_{\sigma \in S_n} \varepsilon(\sigma)v_{\sigma(1)} \ldots v_{\sigma(n)}. \]

It is immediate that \( \pi \circ N = \text{id}_S \), where here and subsequently, we abbreviate \( \bar{S}(V) \) and \( S(V) \) to \( S \) in subscripts. Using Lemma 15, we compute

\[ \text{See this equation above.} \]

This shows that \( \text{im}(N) \subset \bar{T}(V) \) is a subcoalgebra and induces a coproduct on \( \bar{S}(V) \cong \text{im}(N) \). As a subcoalgebra of a coassociative coalgebra is clearly coassociative itself, \( \bar{S}(V) \) becomes a coaugmented coalgebra with the coproduct \( \Delta_S : S(V) \rightarrow S(V) \otimes S(V) \) given by

\[ \Delta_S(v_1 \ldots v_n) = \sum \sum_{\tau \in Sh_{n-i}} \varepsilon(\tau)(v_{\tau(1)} \ldots v_{\tau(i)}) \otimes (v_{\tau(i+1)} \ldots v_{\tau(n)}), \quad (18) \]

It is immediate that \( S(V) \) is comultiplicative, as it is a subcoalgebra of \( T(V) \). We claim that \( S(V) \) is even comultiplicative. Indeed, let \( \sigma_i \in \mathfrak{S}_n \) be for \( 0 \leq i \leq n \) the permutation given by \( (\sigma(1), \ldots, \sigma(n)) = (i+1, \ldots, n, i, \ldots) \). We then have \( \text{Sh}_{n-i} : \sigma_i = \text{Sh}_{n-i} \) and therefore

\[ \tau_{S,S} \circ \Delta_S(v_1 \ldots v_n) \]
\[ = \sum_{i=0}^{n} \sum_{\tau \in Sh_{n-i}} \varepsilon(\tau(\sigma_i))(v_{\tau(\sigma_i)(1)} \ldots v_{\tau(\sigma_i)(n-i)}) \]
\[ \otimes (v_{\tau(\sigma_i)(n-i+1)} \ldots v_{\tau(\sigma_i)(n)}) \]
\[ = \sum_{i=0}^{n} \sum_{\tau \in Sh_{n-i}} \varepsilon(\tau)v_{\tau(1)} \ldots v_{\tau(i)} \]
\[ \otimes (v_{\tau(i+1)} \ldots v_{\tau(n)}) \]
\[ = \Delta_S(v_1 \ldots v_n). \]

**Proposition 18.** Let \( (C, \Delta) \) be a comultiplicative comonoidal coalgebra and \( f : C \rightarrow V \) a degree preserving linear map. There is a unique homomorphism of coaugmented coalgebras \( f : C \rightarrow \bar{S}(V) \) such that \( f = \text{pr}_V \circ \hat{f} \).

**Proof.** As in the proof of Proposition 17, it suffices to show that there is a unique homomorphism of coalgebras \( \bar{f} : C \rightarrow \bar{S}(V) \) satisfying \( \text{pr}_V \circ \bar{f} = f \). Recall that \( \bar{T}(f) \circ \sum_{n} \Delta^n \) is the unique coalgebra homomorphism \( C \rightarrow \bar{T}(V) \) extending \( f \). For \( 0 \leq i \leq n-1 \), we have

\[ (\text{id}_V \otimes \tau_{C,C} \otimes \text{id}_V)\bar{T}(f) \circ \Delta^n \]
\[ = \bar{T}(f)(\text{id}_C \otimes \tau_{C,C} \otimes \text{id}_C)^{(n-1)}(\Delta)^{n} \]
\[ = \bar{T}(f)(\text{id}_C \otimes \tau_{C,C} \otimes \text{id}_C)^{(n-1)}(\Delta)^{n-1} \]
\[ = \bar{T}(f)(\text{id}_C \otimes \Delta \otimes \text{id}_C)^{(n-1)}(\Delta)^{n-1} \]
\[ = \bar{T}(f) \circ \Delta^{n}. \]

Since \( C \) is comultiplicative. As \( (\text{id}_V \otimes \tau_{C,C} \otimes \text{id}_V)^{(n-i)} = \varepsilon(s_i) \), the image of \( \bar{T}(f) \circ \sum_{n-i} \Delta^n \) is contained in the subspace of \( \bar{T}(V) \) of all symmetric elements, which is \( \text{im}(N) \).
We obtain an induced homomorphism of coalgebras

\[
\tilde{f} = \pi \circ T(f) \circ \sum_{n=0}^{\infty} A^n : \mathcal{C} \to \mathcal{S}(V),
\]

\[
x \mapsto \frac{1}{n!} \sum_{m=1}^{n} f(x(1)) \vee \ldots \vee f(x(n))
\]

with \( \text{pr}_V \circ \tilde{f} = f \). Similarly, a coalgebra homomorphism \( \tilde{f} : \mathcal{C} \to \mathcal{S}(V) \) gives rise to a coalgebra homomorphism \( N \circ \tilde{f} : \mathcal{C} \to \mathcal{T}(V) \) that is uniquely determined by \( \text{pr}_V \circ N \circ \tilde{f} = f \) by Proposition 17. As \( N \) is injective, this shows uniqueness of \( \tilde{f} \).

**Example 19.** A linear degree preserving map \( f : V \to W \) can be extended by zero to a linear map \( \mathcal{S}(V) \to \mathcal{W} \). The induced homomorphism of coaugmented coalgebras \( \mathcal{S}(V) \to \mathcal{S}(W) \) is denoted by \( \mathcal{S}(f) \) and is given by \( \mathcal{S}(f)(v_1 \vee \ldots \vee v_n) = f(v_1) \vee \ldots \vee f(v_n) \).

### 2.4.3 Comodules and coderivations

Let \((C, \Delta)\) be a coassociative coalgebra. A left comodule over \(C\) is a graded vector space \(M\) together with a degree preserving linear map \(\Delta_l : M \to C \otimes M\) satisfying

\[
(\Delta \otimes \text{id}_M)\Delta_l = (\text{id}_C \otimes \Delta_l)\Delta_l.
\]  

(20)

Similarly, a right comodule over \(C\) is a graded vector space \(M\) together with a degree preserving linear map \(\Delta_r : M \to M \otimes C\) such that

\[
(\text{id}_M \otimes \Delta)\Delta_r = (\Delta_r \otimes \text{id}_C)\Delta_r.
\]  

(21)

If \(M\) is both a left and a right comodule over \(C\) and if the compatibility relation

\[
(\Delta_l \otimes \text{id}_C)\Delta_r = (\text{id}_C \otimes \Delta_r)\Delta_l
\]  

is satisfied, \(M\) is called a (bi)comodule over \(C\). Given such \(M\), we define a coderivation of degree \(p\) to be a homogeneous linear map \(d : M \to C\) of degree \(p\) such that

\[
\Delta \circ d = (d \otimes \text{id}_C)\Delta_r + (\text{id}_C \otimes d)\Delta_l.
\]  

(22)

We denote the vector space of all these maps by \(\text{Coder}_p(M, C)\) and by \(\text{Coder}(M, C)\) the graded vector space \(\bigoplus_{p \in \mathbb{Z}} \text{Coder}_p(M, C)\).

Let \((C, \Delta_C)\) and \((D, \Delta_D)\) be coassociative coalgebras and \(f : D \to C\) a cohomomorphism. Then \(\Delta_r := (\text{id}_D \otimes f)\Delta_D\) and \(\Delta_l := (f \otimes \text{id}_D)\Delta_D\) make \(D\) into a comodule over \(C\). In particular, \(C\) is a comodule over itself and we abbreviate \(\text{Coder}(C, C)\) to \(\text{Coder}(C)\). If \(C\) and \(D\) are coaugmented and if \(f\) is a homomorphism of coaugmented coalgebras, the comodule structure is compatible with the counit in the sense that the diagram

\[
\begin{array}{ccc}
C \otimes D & \xrightarrow{\Delta_l} & D \otimes C \\
\varepsilon \otimes \text{id}_D & \downarrow & \Delta_r \\
\text{id}_D \otimes \varepsilon & \downarrow & \text{id}_D \otimes \text{id}_C
\end{array}
\]

(24)

commutes, where \(\varepsilon\) is the counit on \(C\). Observe that \(\tilde{f} : \mathcal{D} \to \mathcal{C}\) then makes \(\mathcal{D}\) into a comodule over \(\mathcal{C}\). The following proposition relates elements in \(\text{Coder}(D, C)\) to coderivations \(d : D \to C\) that satisfy \(d(1) = 0\); the latter is called a coderivation of coaugmented coalgebras.

**Proposition 20.** Let \(f : D \to C\) be a homomorphism of coaugmented coalgebras. There is a one-to-one correspondence between coderivations \(d : D \to C\) satisfying \(d(1) = 0\) and coderivations \(\tilde{d} : \mathcal{D} \to \mathcal{C}\) given by \(d = \tilde{d} \oplus 0 : \mathcal{D} \oplus k \to \mathcal{C} \oplus k\).

**Proof.** Given a linear map \(\tilde{d} \in \text{Coder}(\mathcal{D}, \mathcal{C})\), one easily checks that \(\tilde{d}\) is a coderivation if and only if \(d \oplus 0 : D \oplus k \to \mathcal{C} \oplus k\) is. It then suffices to show that each coderivation \(d : D \to C\) with \(d(1) = 0\) is of this form. Let \(\varepsilon\) be the counit of \(C\) and \(\mu_k : k \otimes k \to k\) the multiplication on \(k\). From (13), (23) and the compatibility with the counit (24) it then follows that

\[
\varepsilon \circ d = \mu_k((\varepsilon \otimes \varepsilon)\Delta_C \circ d)
\]

\[
= \mu_k((\varepsilon \otimes (\varepsilon \circ d))\Delta_l + \mu_k((\varepsilon \circ d) \otimes \varepsilon)\Delta_r)
\]

\[
= 2(\varepsilon \circ d),
\]

which shows that \(d(D) \subset \mathcal{C}\). Hence, \(d\) decomposes as \(d \oplus 0 : \mathcal{D} \oplus k \to \mathcal{C} \oplus k\).

For a coassociative coalgebra \(C\), we call an element \(d \in \text{Coder}(C)\) of degree one with \(d^2 = 0\) a coderiferential on \(C\). We then call the pair \((C, d)\) a differential graded coassociative coalgebra (DGCA for short). If \(C\) is coaugmented and \(d(1) = 0\), we call \((C, d)\) a coaugmented DGCA. A homomorphism of DGCs is then a coalgebra homomorphism that is also a homomorphism of DG vector spaces; homomorphisms of coaugmented DGCS are defined accordingly. From Proposition 20 and Section 2.4.1, we then obtain an equivalence between the categories of DGCS and coaugmented DGCS.

**Proposition 21.** Let \(C\) be a coassociative coalgebra. Then \(\text{Coder}(C) \subset \mathfrak{gl}(C)\) is closed under the graded commutator. Also, if \(f : D \to C\) is a homomorphism of coassociative coalgebras, \(d \in \text{Coder}(C)\) and \(d' \in \text{Coder}(D)\), then \(f \circ d' \circ f \in \text{Coder}(D, C)\).

**Proof.** Both parts of the proposition are straightforward computations which are left to the reader.

**Theorem 22.** Let \(D\) be a cocommutative coaugmented coalgebra and \(f : D \to \mathcal{S}(V)\) a homomorphism of coaugmented coalgebras. The linear map

\[
\text{Coder}(D, \mathcal{S}(V)) \to \text{Hom}(D, V), \quad d \mapsto \text{pr}_V \circ d
\]

is then an isomorphism. Its inverse is given by

\[
\text{Hom}(D, V) \to \text{Coder}(D, \mathcal{S}(V)), \quad \lambda \mapsto \mu_S(\lambda \otimes f)\Delta_D,
\]

where \(\mu_S : \mathcal{S}(V) \otimes \mathcal{S}(V) \to \mathcal{S}(V)\) denotes the multiplication on \(\mathcal{S}(V)\) and \(\Delta_D\) the coproduct on \(D\).

It is immediate that \(d(1) = 0\) if and only if \(\lambda = \text{pr}_V \circ d\) vanishes on \(k\). Together with Proposition 20, this shows \(\text{Coder}(\mathcal{D}, \mathcal{S}(V)) \cong \text{Hom}(\mathcal{D}, \mathcal{V})\).
\[ \Delta_S(v_1) \ldots \Delta_S(v_{n+1}) = \Delta_S(v_1 \vee \ldots \vee v_n) \Delta_S(v_{n+1}) \]

\[ = \sum_{i=0}^{n} \sum_{\tau \in \text{Sh}_{n-i-1}^i} (\varepsilon(\tau)(v_{\tau(1)} \vee \ldots \vee v_{\tau(i)}) \otimes (v_{\tau(i+1)} \vee \ldots \vee v_{\tau(n)})) \left(v_{n+1} \otimes 1 + 1 \otimes v_{n+1}\right) \]

\[ = \sum_{i=0}^{n} \sum_{\sigma \in \text{Sh}_{n-i-1}^i} (-1)^{|v_{n+1}|} \sum_{\tau \in \text{Sh}_{n-i}^{i+1}} \varepsilon(\sigma)(v_{\sigma(1)} \vee \ldots \vee v_{\sigma(i)} \otimes (v_{\sigma(i+1)} \vee \ldots \vee v_{\sigma(n)} \otimes v_{n+1}) \]

\[ + \sum_{i=0}^{n} \sum_{\sigma \in \text{Sh}_{n-i-1}^i} \varepsilon(\sigma)(v_{\sigma(1)} \ldots \vee v_{\sigma(i)} \otimes (v_{\sigma(i+1)} \ldots \vee v_{\sigma(n)} \otimes v_{n+1}) \]

\[ = \sum_{i=0}^{n+1} \sum_{\sigma \in \text{Sh}_{n+1-i}^{i+1}} \varepsilon(\sigma)(v_{\sigma(1)} \ldots \vee v_{\sigma(i)} \otimes (v_{\sigma(i+1)} \ldots \vee v_{\sigma(n)} \otimes v_{n+1}) \]

\[ = \Delta_S(v_1 \vee \ldots \vee v_{n+1}). \]

The first part of Theorem 22 actually holds for a broader class of comodules over \( S(V) \) (see for example [8], Lemma 2.4); the inverse formula \( d = \mu_S(\alpha \otimes \text{id}_M)\Delta_s \) then continues to hold for comodules \( M \) in which \( \tau_{M,S} \circ \Delta = \Delta_t \).

**Remark 23.** For \( 1 \leq i \leq n-1 \) and \( \tau \in \text{Sh}_{n+1-n-i}^i \) either \( \tau(i) = n \) or \( \tau(n) = n \). In the first case, there is a unique \( \sigma \in \text{Sh}_{n-1-n-i}^i \) such that \( (\tau(1), \ldots, \tau(n)) = (\sigma(1), \ldots, \sigma(i-1), n, \sigma(i), \ldots, \sigma(n-1)) \), while in the second case \( (\tau(1), \ldots, \tau(n)) = (\sigma(1), \ldots, \sigma(n-1), n) \) for a unique \( \sigma \in \text{Sh}_{n-1-n-i}^{i-1} \). This yields a bijection \( \text{Sh}_{n-1-n-i}^i \cong \text{Sh}_{n-1-n-i}^{i-1} \). By setting \( \text{Sh}_{n-1-n-i}^i = \emptyset \), this also holds for \( i = 0, n \).

**Lemma 24.** The map \( \Delta_S : S(V) \to S(V) \otimes S(V) \) is a homomorphism of graded algebras.

**Proof.** We show by induction over \( n \in \mathbb{N} \) that
\[ \Delta_S(v_1 \vee \ldots \vee v_n) = \Delta_S(v_1) \ldots \Delta_S(v_n). \quad (25) \]

For \( n = 1 \) there is nothing to do. Assume that (25) holds for \( n \geq 1 \). We compute

\textit{See this equation above.}

In the fourth equality, we shifted the summation index of the first sum and used Remark 23.

**Proof of Theorem 22.** Let \( d : D \to S(V) \) be a coderivation, that is
\[ \Delta_S \circ d = (d \otimes f + f \otimes d) \Delta_D. \]

Inductively, we then get
\[ \Delta^n_S \circ d = \sum_{k=0}^{n} (f_{(k)} \otimes d) \otimes (f \otimes (n-k)) \Lambda^n_D. \]

For \( n \in \mathbb{N} \), let \( \pi_n : T^n(V) \to S^n(V) \) be the linear map defined by
\[ \pi_n(v_1 \otimes \ldots \otimes v_n) = \frac{1}{n!} v_1 \vee \ldots \vee v_n. \]

From \( S(V) \) being a subcoalgebra of \( T(V) \) and (17), it follows that \( \pi_{n+1} \circ \text{pr}_V \otimes (n+1) \Delta^n_S = \text{pr}_{S^{n+1}}(V) \). We then have
\[ \text{pr}_{S^{n+1}}(V) \circ d = \pi_{n+1} \circ \text{pr}_V \otimes (n+1) \]

\[ \quad \circ \sum_{k=0}^{n} (f_{(k)} \otimes d) \otimes (f \otimes (n-k)) \Delta^n_D = \pi_{n+1} \circ \sum_{k=0}^{n} ((f \otimes f_{(k)} \otimes (f \circ d) \]

\[ \quad \otimes (f \otimes f_{(k)} \otimes (f \circ d)) \Delta^n_D. \]

As this holds for all \( n \in \mathbb{N} \) and as \( \text{pr}_V \circ d = 0 \) by the same computation as in the proof of Proposition 20, \( d \) is completely determined by \( \text{pr}_V \circ d \).

What is left is to show that given \( \lambda \in \text{Hom}(D,V) \) homogeneous, \( d := \mu_S(\lambda \otimes f) \Delta_D \) is a coderivation with \( \text{pr}_V \circ d = \lambda \). While the latter holds by construction, we compute for \( x \in D \) homogeneous

\textit{See this equation next page}

where we used Lemma 24 in the first and cocommutativity of \( D \) in the fifth equality.

**2.5 Dual spaces**

The graded vector space \( V^* := \text{Hom}(V,k) \) is called the dual space of \( V \). By degree reasons, \( (V^*)_{(k)} := \text{Hom}_k(V,k) = \text{Hom}(V,k) \). For \( f \in \text{Hom}_k(V,W) \), the linear map \( f^* \in \text{Hom}_k(W^*,V^*) \) is defined by \( f^*(\varphi) = (-1)^{|\varphi||f|} \varphi \circ f \) for \( \varphi \in W^* \) homogeneous. Note that
We then have
\[ (\Delta \circ d)(x) = \sum \Delta S(\lambda(x(1))) \vee \Delta S(f(x(2))) \]
\[ = \sum (\lambda(x(1)) \otimes 1 + 1 \otimes \lambda(x(1))) \vee (f \otimes f)(\Delta D(x(2))) \]
\[ = \sum (\lambda(x(1)) \otimes 1 + 1 \otimes \lambda(x(1)))(f(x(2)) \otimes f(x(3))) \]
\[ = \sum (\lambda(x(1)) \vee f(x(2))) \otimes f(x(3)) + (-1)^{(\lambda + |x(1)|)|x(2)|} f(x(2)) \otimes (\lambda(x(1)) \vee f(x(3))) \]
\[ = \sum (\mu_S(\lambda \otimes f) \otimes f + f \otimes \mu_S(\lambda \otimes f))(x(1) \otimes x(2) \otimes x(3)) \]
\[ = (\mu_S(\lambda \otimes f)\Delta D \otimes f + f \otimes \mu_S(\lambda \otimes f)\Delta D)(\Delta D(x)), \]

\[ id_V^* = \text{id}_{V^*} \] and if \( g \) is a homogeneous linear map with domain \( W \), we have \( (g \circ f)^* = (-1)^{|f||g|} f^* \circ g^* \).

We say that \( V \) is of finite type if \( V_k \) is finite-dimensional for all \( k \in \mathbb{Z} \). Note that if \( V \) is of finite type, the canonical inclusion \( V \hookrightarrow V^{**} \) is an isomorphism.

If \( V_k = 0 \) for \( k > 0 \), then \( V \) is called \( \mathbb{Z}_{<0} \)-graded. Notions as \( \mathbb{Z}_{<0} \)-graded or \( \mathbb{Z}_{>0} \)-graded are defined accordingly. In the following, we denote \( V^{\otimes n} \) as \( T^n(V) \) for better readability.

**Proposition 25.** If \( V \) is of finite type and if for all \( k \in \mathbb{Z} \) the decomposition
\[ T^n(V)_k = \bigoplus_{i_1 + \ldots + i_n = k} (V_{i_1} \otimes \ldots \otimes V_{i_n}) \]
has only finitely many non-trivial summands, then the canonical inclusion \( T^n(V^*) \hookrightarrow T^n(V^*) \) is an isomorphism.

**Proof.** It is well-known that for finite-dimensional (ungraded) vector spaces \( V_1, \ldots, V_n \), the canonical inclusion \( V_1 \otimes \ldots \otimes V_n \hookrightarrow (V_1 \otimes \ldots \otimes V_n)^* \) is an isomorphism. We then have
\[ (T^n(V^*)^*)_{i_1 + \ldots + i_n = k} = \bigoplus_{i_1 + \ldots + i_n = k} (V^*)_{i_1} \otimes \ldots \otimes (V^*)_{i_n} \]
\[ = T^n(V^*)_k. \]

**Remark 26.** If \( V \) is of finite type and \( \mathbb{Z}_{<0} \)-graded, \( V^* \) is also of finite type and \( \mathbb{Z}_{>0} \)-graded and they both satisfy the hypothesis of Proposition 25. It is then easy to see that \( \Delta: V \rightarrow V \otimes V \) makes \( V \) into a graded coassociative/co-commutative coalgebra if and only if \( \Delta^*: V^* \otimes V^* \cong (V \otimes V)^* \rightarrow V^* \) makes \( V^* \) into an associative/commutative algebra. A linear map \( d: V \rightarrow V \) is then a coderivation of \((V, \Delta)\) if and only if \( -d^* \) is a derivation of \((V^*, \Delta^*)\). The map \( gl(V) \rightarrow gl(V^*) \), \( f \mapsto -f^* \) preserves the graded commutator and therefore restricts to an isomorphism of graded Lie algebras \( \text{Coder}(V) \cong \text{Der}(V^*) \).

**Corollary 27.** If \( V \) is of finite type and \( \mathbb{Z}_{<0} \)-graded, the canonical inclusion \( T(V^*) \hookrightarrow T(V^*) \) is an isomorphism.

**Proof.** Note that for all \( k \in \mathbb{Z} \) and \( n > k \), we have \( T^n(V^*)^{-k} = 0 \). Then
\[ (T^n(V^*)^{-k})^* \cong \bigoplus_{n \geq k} (T^n(V^*)^*)_k \]
\[ \cong \bigoplus_{n \geq k} T^n(V^*)_k = T^n(V^*). \]

**Lemma 28.** Let \( \xi \in T^n(V^*) \subset T^n(V^*) \) and \( \sigma \in S_n \). Then
\[ \hat{\epsilon}(\sigma)\xi = \xi \circ \hat{\epsilon}(\sigma^{-1}). \]

**Proof.** It suffices to show this for \( \sigma = s_1, \ldots, s_n \), in which case it is an easy computation.

**Proposition 29.** If \( V \) is of finite type and \( \mathbb{Z}_{<0} \)-graded, \( S(V)^* \cong S(V^*) \). Under this identification, \( \Delta_S^* \) is the usual multiplication on \( S(V^*) \).

**Proof.** Fix \( n \geq 0 \). By Lemma 28, the isomorphism \( T^n(V^*) \cong T^n(V^*)^* \) maps the subspace of symmetric elements in \( T^n(V^*) \) onto the space of symmetric linear maps \( V^\otimes_n \rightarrow k \). While the latter is isomorphic to \( \text{Hom}(S^n(V), k) \cong S^n(V^*) \) by Remark 2, the former is isomorphic to \( S^n(V^*) \) via the linear map
\[ S^n(V^*) \rightarrow T^n(V^*), \]
\[ v_1 \vee \ldots \vee v_n \mapsto \sum_{\sigma \in S_n} \xi(\sigma)v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}. \]

This yields an isomorphism \( S^n(V^*) \cong S^n(V^*) \). With a similar reasoning as in the proof of Corollary 27, one obtains \( S(V^*) \cong S(V)^* \). It is then a straightforward computation to show that \( \Delta_S^* \) is indeed the usual multiplication on \( S(V^*) \).

### 3 \( L_\infty \)-algebras

We start this section with a theorem from [5] that characterises certain \( L_\infty \)-algebras using Lie algebra cohomology; later, we seek to generalise it in the context of \( L_\infty \)-algebra cohomology. After that, we discuss different characterisations of \( L_\infty \)-structures using the key results of Section 2. Different points of view naturally lead to different notions of homomorphisms between \( L_\infty \)-algebras; we will finish the section with a comparison of those. For this, we will mostly...
Chevalley–Eilenberg (cochain) comorphism of Lie algebras $g$

For an (ungraded) Lie algebra $L$ of degree reasons and the defect of the Jacobi identity in comparison with equations may be summarized by saying that the generalized Jacobi identity

$$\sum_{\sigma \in \text{Sh}_{n-1}} (-1)^{|\sigma|} \chi(\sigma) l_2(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) + l_1(l_3(x_1, x_2, x_3)) + l_3(l_1(x_1, x_2, x_3))$$

holds for all $n \geq 1$ and $x_1, \ldots, x_n \in L$ homogeneous. Then call the set $\{l_k \mid 1 \leq k < \infty\}$ an $L_\infty$-structure on $L$.

Writing out (27) for $n = 1, 2, 3$ yields

$$\text{See this equation above}$$

for all $x_1, x_2, x_3 \in L$ homogeneous. While the first two equations may be summarized by saying that $l_1$ is a differential on the (non-associative) graded algebra $(L, l_2)$, a comparison with (11) shows that the third one describes the defect of the Jacobi identity in $(L, l_2)$. In particular, an $L_\infty$-algebra with $l_k = 0$ for $k \geq 3$ is nothing else than a DGLA.

If $L$ is concentrated in degree zero, $l_k = 0$ for $k \neq 2$ by degree reasons and $(L, l_2)$ is an (ungraded) Lie algebra.

Definition 30. An $L_\infty$-algebra is a graded vector space $L$ together with antisymmetric linear maps $l_k: L^\otimes k \to L$ called (higher) brackets of degree $|l_k| = 2 - k$ for $1 \leq k < \infty$. The generalized Jacobi identity

$$\sum_{i+j=n+1} \sum_{\sigma \in \text{Sh}_{n-1}} (-1)^{|\sigma|} \chi(\sigma) l_2(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) + l_1(l_3(x_1, x_2, x_3)) + l_3(l_1(x_1, x_2, x_3))$$

is concentrated in degree zero, as is the case for $\sigma$-structures.

Definition 31. Let $L$ and $L'$ be $L_\infty$-algebras with $L_\infty$-structures $\{l_k\}_{k \in \mathbb{N}}$ and $\{l'_k\}_{k \in \mathbb{N}}$, respectively. A strict $L_\infty$-algebra homomorphism is a degree preserving linear map $f: L \to L'$ satisfying

$$f \circ l_k = l'_k \circ f^\otimes k$$

for all $1 \leq k < \infty$.

These homomorphisms are strict in the sense that they strictly preserve all brackets. A different characterisation of $L_\infty$-algebras will later lead to a more general notion of $L_\infty$-algebra homomorphisms.

3.1 Characterisation via Lie algebra cohomology

For an (ungraded) Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$, a representation of $\mathfrak{g}$ on an (ungraded) vector space $V$ is a homomorphism of Lie algebras $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$. Given such a $\rho$, the Lie algebra cohomology with values in $V$ is the cohomology of the Chevalley–Eilenberg (cochain) complex $(\bigoplus_{n \geq 0} \text{Hom}((\wedge^n \mathfrak{g}, V), \delta))$, where for an antisymmetric linear map $\omega: \mathfrak{g}^\otimes n \to V$, we define $\delta \omega$ by

$$\delta \omega(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} \rho(x_i)(\omega(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}))$$

for $x_1, \ldots, x_{n+1} \in \mathfrak{g}$. In the sums above, elements with $\hat{\cdot}$ are to be omitted.

Theorem 32 ([5], Theorem 55). There is for each $n \geq 1$ a one-to-one correspondence between $L_\infty$-algebras $L$ such that $l_k = 0$ for $k \neq -n, 0$ and $l_1 = 0$ and quadruples $(\mathfrak{g}, V, \rho, l_{n+2})$ consisting of a Lie algebra $\mathfrak{g}$, a representation $\rho$ of $\mathfrak{g}$ on a vector space $V$ and an $(n+2)$-cocycle $l_{n+2}$.

Sketch of proof. For an $L_\infty$-algebra $L = L_0 \otimes L_n$ with $l_1 = 0$, all brackets except for $l_2$ and $l_{n+2}$ have to vanish by degree reasons. Also, $l_2$ has to vanish on $\wedge^2 L_{n-1}$ and $l_{n+2}$ can only be non-trivial on $\wedge^{n+2} L_0$ with image in $L_{-n}$. Using (8), we can decompose $l_2$ into linear maps $[\cdot, \cdot]: \wedge^2 L_0 \to L_0$ and $\rho: L_0 \otimes L_{-n} \to L_{-n}$. It is then a matter of computation to show that $l_2$ and $l_{n+2}$ satisfying (27) amounts to $(L_0, [\cdot, \cdot])$ being a Lie algebra, $\rho$ being a representation of $L_0$ on $L_{-n}$ and $l_{n+2}$ being a cocycle.

3.2 Symmetric brackets and codifferentials

Recall that by Corollary 4, an antisymmetric map $l_k: L^\otimes k \to L$ of degree $2 - k$ is equivalent to a symmetric degree one map $\lambda_k: S^k L[1] \to L[1]$ such that

$$l_k = \uparrow \circ \lambda_k \circ \downarrow^\otimes k.$$ 

If we now rewrite (27) in terms of the maps $\lambda_k$, we obtain a characterisation of $L_\infty$-structures on $L$ in terms of symmetric brackets. Note that for fixed $n$, we can write (27) as

$$\sum_{i+j=n+1} \sum_{\sigma \in \text{Sh}_{n-1}} (-1)^{|\sigma|} l_2(l_i \otimes \id_{L}^\otimes (-1)^{|\sigma|}) \chi(\sigma) = 0$$

As $\downarrow$ and $\uparrow^\otimes n$ are isomorphisms, this is equivalent to

$$\text{See this equation next page}$$

where we used (2) and (4). We have proved the following.
Theorem 36. An $L_\infty$-structure on the graded vector space $L$ is equivalent to a codifferential $d \in \text{Coder}(S(L[1]))$ with $d(1) = 0$.

In this case, we also refer to the pair $(S(L[1]), d)$ as an $L_\infty$-algebra.

Proof. Let $\lambda : S(L[1]) \to L[1]$ be of degree one and $d = \mu_S(\lambda \otimes 1 L)\Delta_S$ be the unique coderivation extending $\lambda$ in the sense of Theorem 22. As $d^2 = \frac{1}{2}[d, d]$ is a coderivation of $S(L[1])$ by Proposition 21, we have by Theorem 22 that $d^2 = 0$ if and only if

$$0 = \text{pr}_{L[1]} \circ d^2 = \lambda \circ \mu_S(\lambda \otimes 1 L)\Delta_S.$$

Corollary 37. If $L$ is of finite type and $\mathbb{Z}_{<0}$-graded, an $L_\infty$-structure on $L$ is equivalent to a differential on the graded algebra $S(L[1]^*)$. Explicitly, consider $d_{CE} = -d^*$ for $d$ as in Theorem 36.

Proof. See Remark 26 and note that as $L$ is $\mathbb{Z}_{<0}$-graded, each $d \in \text{Coder}(S(L[1]))$ of degree one vanishes on $k$ by degree reasons.

### 3.3 Weak homomorphisms

Let $(S(L[1]), d)$ and $(S(L'[1]), d')$ be $L_\infty$-algebras, $\lambda = \text{pr}_{L[1]} \circ d$ and $\lambda' = \text{pr}_{L'[1]} \circ d'$. The characterisation of $L_\infty$-structures as codifferentials on the symmetric coalgebra leads to another notion of homomorphisms of $L_\infty$-algebras, namely as homomorphisms of (coaugmented) DGCs.

Definition 38. A (weak) homomorphism of $L_\infty$-algebras between $L$ and $L'$ is a homomorphism of coaugmented DGCs $f : S(L[1]) \to S(L'[1])$.

Remark 39. By Proposition 21 and Theorem 22, a homomorphism of coaugmented coalgebras $f : S(L[1]) \to S(L'[1])$ is a homomorphism of $L_\infty$-algebras if and only if

$$(\text{pr}_{L[1]} \circ f) \circ d = \lambda' \circ f.$$
With now two different notions of $L_\infty$-algebra homomorphisms at hand, it is reasonable to ask if there is a connection between them. As commented in [8], Remark 5.3, strict homomorphisms are essentially the weak homomorphisms that preserve the exterior degree.

**Lemma 41.** Let $g: L[1] \to L'[1]$ be a linear degree preserving map. Then $g$ is a strict $L_\infty$-algebra homomorphism if and only if $S(g)$ is a weak one.

**Proof.** Observe that $\text{pr}_{L'[1]} \circ S(g) \circ d = g \circ \text{pr}_{L[1]} \circ d = g \circ \lambda$.

**Lemma 42.** A homomorphism of coalgebras $f: S(L[1]) \to S(L'[1])$ preserves the exterior degree if and only if $f = S(g)$ for a linear degree preserving map $g: L[1] \to L'[1]$.

**Proof.** Assume that $f: S(L[1]) \to S(L'[1])$ is a homomorphism of coalgebras such that $f(S^n(L[1])) \subset S^n(L'[1])$ for all $n$ and let $g := \text{pr}_{L'[1]} \circ f|_{L[1]}$. Then $\text{pr}_{L'[1]} \circ f = g \circ \text{pr}_{L[1]} = \text{pr}_{L'[1]} \circ S(g)$. Hence, $f = S(g)$ by Proposition 18.

**Proposition 43.** Let $f: S(L[1]) \to S(L'[1])$ be a (weak) $L_\infty$-algebra homomorphism. Then $f$ preserves the exterior degree if and only if $f = S(g)$ for a strict $L_\infty$-algebra homomorphism $g$.

**Proof.** Combine Lemma 41 and Lemma 42.

From this it follows for example that all (weak) $L_\infty$-algebra homomorphisms between Lie algebras are induced by Lie algebra homomorphisms.

### 4 Representations (up to homotopy)

While representations (up to homotopy) of $L_\infty$-algebras are often defined in terms of antisymmetric maps, we start with a definition that keeps the symmetric point of view of the last section. While it is a straightforward computation to show equivalence between these definitions, it is convenient to save this for Section 5.1. We then show that representations (up to homotopy) are nothing else than weak $L_\infty$-algebra homomorphisms into $\mathfrak{gl}(V)$ for a DG vector space $V$, a characterisation due to Lada and Markl [8]. In [3], representations (up to homotopy) were described (under some finiteness assumptions) as differentials on $S(L[1]^*) \otimes V$. We discuss this point of view in the second half of this section, which also leads us to $L_\infty$-algebra cohomology.

From now on, $L$ denotes an $L_\infty$-algebra with $L_\infty$-structure $\{l_k | 1 \leq k < \infty\}$ and $\lambda$ and $d$ are as in Corollary 34 and Theorem 36, respectively.

**Definition 44.** A representation (up to homotopy) of $L$ on $V$ is a linear map $\rho: S(L[1]) \otimes V \to V$ of degree one that satisfies

$$\rho(d \otimes \text{id}_V) + \rho(\text{id}_S \otimes \rho)(\Delta_S \otimes \text{id}_V) = 0. \quad (33)$$

#### 4.1 Representations as (weak) homomorphisms

We prove the following version of ([8], Theorem 5.2).

**Theorem 45.** There is a one-to-one correspondence between representations of $L$ on $V$ and pairs $(\partial, f)$, where $\partial$ is a differential on $V$ and $f: S(L[1]) \to S(\mathfrak{gl}(V)[1])$ a homomorphism of $L_\infty$-algebras. Here, $\mathfrak{gl}(V)$ carries the DGLA structure induced by $\partial$, see Example 12.

One should therefore really think of $L$ being represented on a DG vector space. The following lemma characterises $L_\infty$-algebra homomorphisms into DGLAs and is a symmetric version of ([8], Definition 5.2).

**Lemma 46.** Let $(L', l_2', l_3')$ be a DGLA and $\lambda = \lambda_1 + \lambda_2$ be the corresponding linear degree one map $S(L'[1]) \to L'[1]$. For $f: S(L[1]) \to L'[1]$ a linear degree preserving map, the induced homomorphism of coalgebras $\tilde{f}: S(L[1]) \to S(L'[1])$ is a homomorphism of $L_\infty$-algebras if and only if

$$f \circ \partial = \lambda_1 \circ f + \frac{1}{2} \lambda_2' (f \otimes f) \Delta_S. \quad (34)$$

This is the case if and only if the linear degree one map $\rho: S(L[1]) \to L'$ defined by $f = (-1)^{|x|} \rho |_{L[1]}$ satisfies

$$\rho \circ \partial + l_1' \circ \rho + \frac{1}{2} l_2' (\rho \otimes \rho) \Delta_S = 0. \quad (35)$$

**Proof.** The first part follows immediately from Remark 39 and the explicit construction of $f$ (see Proposition 18). It is straightforward to check that for $x \in S(L[1])$ homogeneous,

$$f(d(x)) = (-1)^{|x|} \rho(d(x)),$$

$$\lambda_1(f(x)) = (-1)^{|x|+1} l_1'(\rho(x)),
\lambda_2'(f(x) \Delta_S)(x) = (-1)^{|x|+1} (l_2' (\rho \otimes \rho) \Delta_S)(x),$$

from which the second part then follows.

**Proof of Theorem 45:** As $\text{Hom}(S(L[1]) \otimes V, V) \cong \text{Hom}(S(L[1]), \mathfrak{gl}(V))$, a linear degree one map $\rho: S(L[1]) \otimes V \to V$ can be decomposed into linear degree one maps $\tilde{\rho}: S(L[1]) \to \mathfrak{gl}(V)$ and $\rho_0: k \to \mathfrak{gl}(V)$; the latter being equivalent to the choice of a degree one element $\partial = \rho_0(1_k) \in \mathfrak{gl}(V)$. If we show that under this identification $\rho$ satisfying (33) is equivalent to $\partial^2 = 0$ and $\tilde{\rho}$ satisfying (35), the assertion follows by Lemma 46. For $x \in S(L[1])$ homogeneous,

$$\frac{1}{2} \langle [\cdot, \cdot] (\tilde{\rho} \otimes \rho) \Delta_S \rangle(x) = \frac{1}{2} \sum (-1)^{|z(1)|} |\tilde{\rho}(x(1)), \rho(x(2))|$$

$$\begin{align*}
&= \frac{1}{2} \sum (-1)^{|z(1)|} |\tilde{\rho}(x(1)) \circ \rho(x(2))| \\
&\quad + (-1)^{|x(1)|} |\rho(x(1)) \otimes \rho(x(2))| \circ \rho(x(1)) \\
&= \frac{1}{2} \sum \rho(\text{id}_S \otimes \rho)(x(1) \otimes x(2)) \\
&\quad + (-1)^{|x(1)|} |\rho(x(2)) \otimes x(1), \cdot | \\
&= \rho(\text{id}_S \otimes \rho)(\Delta_S(x), \cdot )
\end{align*}$$
by cocommutativity of \( \overline{S}(L[1]) \) and
\[
[\partial, \tilde{\rho}(x)] = \partial \circ \tilde{\rho}(x) - (1) \tilde{\rho}(x) \circ \partial \\
= \rho_0(1) \circ \tilde{\rho}(x) + (1) \tilde{\rho}(x) \circ \rho_0(1) \\
= \rho(id_S \otimes \rho)(1 \otimes x + x \otimes 1, \cdot).
\]
As \( (\tilde{\rho} \circ d)(x) = (d \otimes id_V)(x, \cdot) \), \( \tilde{\rho} \) satisfying (35) is equivalent to (33) holding on \( \overline{S}(L[1]) \otimes V \). We also have
\[
(\rho(d \otimes \rho)(\Delta_S \otimes \rho_S)) = \rho(1, \rho(1, \cdot)) = \partial^2,
\]
which completes the proof as \( S(L[1]) = \mathbb{k} \otimes \overline{S}(L[1]) \).

**Example 47** (The trivial representation on a DG vector space). Let \( (V, \partial) \) be a DG vector space. There is a trivial strict homomorphism of \( L_\infty \)-algebras \( 0 : L \to gl(V) \). The induced representation \( S(L[1]) \otimes V \to V \) is on \( \mathbb{k} \otimes V \cong V \) given by \( \partial \) and zero elsewhere and is called the **trivial representation** of \( L \) on \( V \). In particular, there is a trivial representation of \( L \) on \( \mathbb{k} \).

**Remark 48**. Let \( \rho \) be a representation of \( L \) on \( V \) and \( (\partial, f) \) as in Theorem 45.

1. Then \( -\partial^* \) is a differential on \( V^* \) and the map \( gl(V) \to gl(V^*), \ g \mapsto -g^* \) is a homomorphism of DGLAs. By composing the corresponding weak homomorphism with \( f \), we obtain an \( L_\infty \)-algebra homomorphism \( S(L[1]) \to S(gl(V^*[1])) \). The induced representation is given by
\[
\rho^\vee : S(L[1]) \otimes V^* \to V^*, \quad x \otimes \xi \mapsto -\rho(x, \cdot)^* \xi
\]
and is called the representation dual to \( \rho \).

2. Fix \( n \in \mathbb{Z} \). Then \( (-1)^n \partial^* \circ \partial \circ \tau^* \) is a differential on \( V[n] \) and
\[
\mathfrak{gl}(V) \to \mathfrak{gl}(V[n]), \ g \mapsto (-1)^n g \otimes \partial \tau^* \n\]
is a DGLA homomorphism. The induced representation of \( L \) on \( V[n] \) is given by
\[
S(L[1]) \otimes V[n] \to V[n], \quad x \otimes \tau^n \mapsto (-1)^{n+n|x|} \rho(x, v).
\]

4.2 Representations as coderivations

Observe that the map \( \Delta_V : = \Delta_S \otimes id_V : S(L[1]) \otimes V \to S(L[1]) \otimes (S(L[1]) \otimes V) \) satisfies
\[
(\Delta_S \otimes id_S \otimes V) \Delta_V = (id_S \otimes \Delta_V) \Delta_V,
\]
which makes \( S(L[1]) \otimes V \) into a left \( S(L[1]) \)-comodule.

**Definition 49**. Let \( d' \in \text{Coder}(S(L[1])) \) be of degree \( p \). A coderivation of \( S(L[1]) \) is a linear map \( D : S(L[1]) \otimes V \) extending \( d' \) is a linear map \( D : S(L[1]) \otimes V \) of degree \( p \) such that
\[
\Delta_V \circ D = (d' \otimes id_{S \otimes V} + id_S \otimes D) \Delta_V.
\]

**Proposition 50** ([6], Proposition 1.5.3, p. 31). Let \( d' \in \text{Coder}(S(L[1])) \) be of degree \( p \). There is a one-to-one correspondence between coderivations \( D \) of \( S(L[1]) \otimes V \) extending \( d' \) and linear maps \( \rho : S(L[1]) \otimes V \) of degree \( p \) given by
\[
D = d' \otimes id_V + (id_S \otimes \rho) \Delta_V, \quad \rho = pr_V \circ D,
\]
where \( pr_V : S(L[1]) \otimes V \to V \) is the projection of \( S(L[1]) \otimes V \) onto \( \mathbb{k} \otimes V \equiv V \).

**Proof.** Let \( D \) be a coderivation of \( S(L[1]) \otimes V \) extending \( d' \). As \( (id_S \otimes pr_V)(\Delta_S \otimes id_V) = id_S \otimes id_V \), we obtain from (37) that
\[
D = (id_S \otimes pr_V) \Delta_V \circ D = (id_S \otimes pr_V)(d' \otimes id_S \otimes id_V + id_S \otimes D) \Delta_V = (d' \otimes id_V)(id_S \otimes pr_V) \Delta_V + (id_S \otimes (pr_V \circ D)) \Delta_V = d' \otimes id_V + (id_S \otimes (pr_V \circ D)) \Delta_V.
\]
This shows that \( D \) is completely determined by \( pr_V \circ D \).

Let conversely \( \rho \in \text{Hom}(S(L[1]) \otimes V, V) \) be of degree \( p \). Using (36) and that \( d' \) is a coderivation, we compute
\[
\Delta_V \circ (id_S \otimes \rho) \Delta_V = (\Delta_S \otimes \rho) \Delta_V = (id_S \otimes \Delta_V) \Delta_V = (id_S \otimes (id_S \otimes \rho)) \Delta_V, \quad \Delta_V \circ (d' \otimes id_V) = ((d' \otimes id_S + id_S \otimes d') \Delta_S) \otimes id_V = (d' \otimes id_S \otimes V + id_S \otimes d') \otimes id_V \Delta_V,
\]
which, combined, show that \( D := d \otimes id_V + (id_S \otimes \rho) \Delta_V \) is a coderivation of \( S(L[1]) \otimes V \) extending \( d \). It is easy to see that then \( pr_V \circ D = \rho \), which completes the proof.

**Corollary 51**. There is a one-to-one correspondence between representations (up to homotopy) of \( L \) on \( V \) and coderivations \( D : S(L[1]) \otimes V \to S(L[1]) \otimes V \) extending \( d \) such that \( D^2 = 0 \).

**Proof.** It is a straightforward computation to check that
\[
D^2 = \frac{1}{2} [D, D] \text{ is a coderivation of } S(L[1]) \otimes V \text{ extending } \frac{1}{2} [d, d] = d^2 = 0. \text{ By Proposition 50, } D^2 = 0 \text{ if and only if } \rho = pr_V \circ D \text{ satisfies}
\]
\[
0 = pr_V \circ D^2 = (d \otimes id_V) + (id_S \otimes \rho) \Delta_S \otimes id_V.
\]

4.3 A first approach to \( L_\infty \)-algebra cohomology

Assume now that the \( L_\infty \)-algebra \( L \) is \( \mathbb{Z}_{\geq 0} \)-graded and of finite type and that \( V \) is either finite-dimensional or of finite type and trivial in the negative degrees. We then have \( S(L[1])^* \cong S(L[1]^*), \text{ } V \cong V^* \) and \( (S(L[1]) \otimes V)^* \cong S(L[1]^*) \otimes V \). Let \( d_{CE} = -d^* \) denote the differential on \( S(L[1]^*) \). The map
\[
(\xi \otimes (\eta \otimes v) \mapsto (\xi \otimes \eta) \otimes v)
\]
makes \( S(L[1]^*) \otimes V \) into a left \( S(L[1]^*) \)-module. Similarly to Definition 49, we call a linear map \( D_{CE} : S(L[1]^*) \otimes \)
\[ V \to S(L[1]^*) \otimes V \] of degree one a derivation of \( S(L[1]^*) \otimes V \) extending \( d_{CE} \) if

\[
D_{CE}(\xi \vee (\eta \otimes v)) = d_{CE}\xi \vee (\eta \otimes v) + (-1)^{[\xi]}(\eta \vee D_{CE}(\eta \otimes v))
\]

holds for all \( \xi, \eta \in S(L[1]^*) \), \( v \in V \) homogeneous.

Note that a representation of \( L \) on \( V \) is equivalent to a representation on \( V^* \) by Remark 48 and \( V \cong V^{**} \). As the notion of a derivation extending \( d_{CE} \) is dual to the one of a coderivation extending \( d \), we get the following dualized version of Corollary 51.

**Proposition 52.** A representation \( \rho \) of \( L \) on \( V \) is equivalent to a derivation \( D_{CE} : S(L[1]^*) \otimes V \to S(L[1]^*) \otimes V \) extending \( d_{CE} \) with \( D_{CE}^2 = 0 \). Explicitly, we have \( D_{CE} = -D^* \), where \( D \) is the coderivation extending \( d \) induced by the dual representation \( \rho^* \).

For a fixed representation \( \rho \) of \( L \) on \( V \), we can then see \( S(L[1]^*) \otimes V \) as our generalized Chevalley–Eilenberg complex with coboundary operator \( D_{CE} \).

### 4.4 A dead-end

This not only provides us with an explicit construction of the coboundary operator from a given representation, but also gives it the additional structure of a derivation extending \( d_{CE} \). Unfortunately, this came at the cost of the finiteness assumptions we imposed on \( L \) on \( V \) at the beginning of Section 4.3. As our goal is to establish a generalisation of Theorem 32 – which does not need such assumptions – in terms of \( L_\infty \)-algebra cohomology, this is not the appropriate framework for our purposes. We can, however, make the following observation.

**Remark 53.** With our finiteness assumptions on \( L \) and \( V \), we have \( S(L[1]^*) \otimes V \cong \text{Hom}(S(L[1]), V) \), where \( \xi \otimes v \in S(L[1]^*) \otimes V \) is identified with the linear map \( S(L[1]) \to V, x \mapsto (-1)^{[\xi]}(\xi(x) \cdot v) \). For \( f \in \text{Hom}(S(L[1]), V) \) homogeneous, one finds that \( D_{CE}f \) is then given by

\[
D_{CE}f = \rho(\id_S \otimes f)\Delta_S - (-1)^{|f|}f \circ d. \tag{38}
\]

One could then simply define \( D_{CE} : \text{Hom}(S(L[1]), V) \to \text{Hom}(S(L[1]), V) \) by (38), even if \( L \) and \( V \) do not meet our finiteness assumptions. Although there is a priori no reason for \( D_{CE}^2 = 0 \) to hold in the general case, a straightforward computation shows that it actually does. While this leaves us with nothing but the formula (38) to work with, it also suggests that there should be another approach to \( L_\infty \)-algebra cohomology that gets by without the need of finiteness assumptions.

In [4], the \( L_\infty \)-algebra cohomology with values in the adjoint representation was introduced in terms of the commutator bracket of coderivations and the isomorphism \( \text{Coder}(S(L[1]), S(L[1])) \cong \text{Hom}(S(L[1]), L[1]) \). In the next section, we extend this approach to arbitrary representations, which leads to a generalisation of Theorem 32 in a rather natural way.

## 5 \( L_\infty \)-algebra cohomology

### 5.1 The Lie bracket on \( \text{Hom}(S(L[1] \oplus V), L[1] \oplus V) \)

Recall from Proposition 21 that \( \text{Coder}(S(L[1])) \) is closed under the graded commutator. Together with Theorem 22, this induces a Lie bracket on \( \text{Hom}(S(L[1]), L[1]) \). Its explicit formula is

\[
[f, g] = \mu_S(g \otimes \id_S)\Delta_S - (-1)^{|f|}|g \circ \mu_S(f \otimes id_S)\Delta_S.
\]

for \( f, g \in \text{Hom}(S(L[1]), L[1]) \) homogeneous.

As \( L_\infty \)-structures correspond to codifferentials with \( d(1) = 0 \) and elements in \( \text{Hom}(S(L[1]), L[1]) \), it is only natural to restrict ourselves to the Lie subalgebra \( \text{Hom}(S(L[1]), L[1]) \). Keeping the \( \text{Hom}(k, L[1]) \) part corresponds to the framework of curved \( L_\infty \)-algebras, which are \( L_\infty \)-algebras that also allow for a 0-ary bracket \( k \to L[1] \).

**Remark 54.** The same construction also makes \( \text{Hom}(S(L[1] \oplus V), L[1] \oplus V) \) into a graded Lie algebra. The decomposition \( S(L[1] \oplus V) \cong S(L[1]) \oplus S(V) \) implies that

\[
\mathfrak{S}(L[1] \oplus V) \cong \mathfrak{S}(L[1]) \oplus \mathfrak{S}(V) \oplus \mathfrak{S}(L[1]) \oplus \mathfrak{S}(V). \tag{40}
\]

We can then consider spaces like \( \mathfrak{S}(L[1], L[1]) \) and \( \text{Hom}(S(L[1]), L[1]) \) as subspaces of \( \text{Hom}(S(L[1] \oplus V), L[1] \oplus V) \) in the obvious way. The inclusion of \( \text{Hom}(S(L[1]), L[1]) \) into \( \text{Hom}(S(L[1] \oplus V), L[1] \oplus V) \) is then easily seen to preserve the Lie bracket.

**Remark 55.** In terms of the Lie bracket on \( \text{Hom}(S(L[1]), L[1]) \), the condition (30) for a linear map \( \lambda : \mathfrak{S}(L[1]) \to L[1] \) of degree one to define an \( L_\infty \)-algebra structure on \( L[1] \) becomes

\[
\frac{1}{2}[\lambda, \lambda] = 0. \tag{41}
\]

By Example 12 and Remark 54, this makes \( \text{Hom}(S(L[1] \oplus V), L[1] \oplus V) \) into a DGLA. Solutions of the Maurer–Cartan equation then induce new \( L_\infty \)-structures on \( L[1] \) by Example 14.

By abuse of notation, we now denote the (co)products on \( \mathfrak{S}(L[1]) \) and \( \text{Hom}(S(L[1]), L[1]) \) by \( \mu_S \) and \( \Delta_S \). This is justified, as they coincide on \( S(L[1]) \subset S(L[1] \oplus V) \).

In (38), \( d = \mu_S(\lambda \otimes id_S)\Delta_S \) and \( \mu_S(id_S \otimes f)\Delta_S = \mu_S(f \otimes id_S)\Delta_S \) due to \( S(L[1]) \) being (co)commutative. The similarity between (38) and (39) suggests to approach \( L_\infty \)-algebra cohomology using the Lie bracket on \( \text{Hom}(\mathfrak{S}(L[1] \oplus V), L[1] \oplus V) \).

**Proposition 56.** Let \( \rho \in \text{Hom}(S(L[1]), V, V) \) be of degree one. Then \( \rho \) is a representation of \( L \) on \( V \) if and only if

\[
\rho \circ \mu_S(\rho \otimes id_S)\Delta_S + \rho \circ \mu_S(\lambda \otimes id_S)\Delta_S = 0, \tag{42}
\]

where \( \lambda \) and \( \rho \) are considered as elements of \( \text{Hom}(\mathfrak{S}(L[1] \oplus V), V, V) \).
Proof. Note that $\rho \circ \mu_S(\rho \otimes \text{id}_S)\Delta_S$ and $\rho \circ \mu_S(\rho \otimes \text{id}_S)\Delta_S$ are only possibly nonzero on $S(L[1]) \otimes V$. For $x_1, \ldots, x_{n-1} \in L[1]$ and $x_n \in V$, a routine computation using Lemma 24 shows that

\[
(\rho \circ \mu_S(\lambda \otimes \text{id}_S)\Delta_S)(x_1 \lor \ldots \lor x_n) = \rho(d(x_1 \lor \ldots \lor x_{n-1}), x_n),
\]

\[
(\rho \circ \mu_S(\lambda \otimes \rho)\Delta_S)(x_1 \lor \ldots \lor x_n) = \rho(\text{id}_S \otimes \rho)(\Delta_S(x_1 \lor \ldots \lor x_{n-1}), x_n).
\]

As again $\mu_S(\rho \otimes \text{id}_S)\Delta_S = \mu_S(\text{id}_S \otimes \rho)\Delta_S$ by (co)commutativity of $S(L[1] \oplus V)$, $\rho$ satisfies (33) if and only if it satisfies (42).

Corollary 57. An element $\rho \in \text{Hom}(S(L[1]) \otimes V, V)$ of degree one is representation of $L$ on $V$ if and only if $(L[1] \oplus V, \rho)$ is an $L_\infty$-algebra.

Proof. We have

\[
\frac{1}{2}[\lambda + \rho, \lambda + \rho] = \frac{1}{2}[\lambda, \lambda] + [\lambda, \rho] + \frac{1}{2}[\rho, \rho] = \rho \circ \mu_S(\lambda \otimes \text{id}_S)\Delta_S + \rho \circ \mu_S(\rho \otimes \text{id}_S)\Delta_S.
\]

Corollary 58. The subspace $\text{Hom}(S(L[1]) \otimes V, V)$ is invariant under the Lie bracket $[-, -]$ and the differential $[\lambda, -]$. Representations (up to homotopy) of $L$ on $V$ are then exactly the Maurer–Cartan elements in $\text{Hom}(S(L[1]) \otimes V, V)$.

By applying Proposition 33 to Corollary 57 and using that a representation on $V$ is equivalent to one on $V[1]$ by Remark 48, we obtain the following.

Proposition 59. A representation of $L$ on $V$ is equivalent to a system of linear maps $\rho_k: \bigwedge^{k-1} L \otimes V \rightarrow V$ of degree $2 - k$ for $k \geq 1$ such that \(\{k + \rho_k: \bigwedge^k (L \oplus V) \rightarrow L \oplus V \mid 1 \leq k < \infty\}\) is an $L_\infty$-structure on $L \otimes V$.

Remark 60. It is easy to see that the generalized Jacobi identity (27) for $\{k + \rho_k \mid 1 \leq k < \infty\}$ has only to be checked on $\bigwedge^{n-1} L \otimes V$ for each $n \geq 1$. Representations of $L_\infty$-algebras are often defined in terms of these equations, see for example [8], Definition 5.1, and [3], Definition 18. Similarly, equation (42) on $S(L[1]) \otimes V$ is easily seen to be the condition imposed on $\rho$ in [3], Definition 19.

For a fixed representation $\rho$ of $L$ on $V$, $[\lambda + \rho, -]$ makes $\text{Hom}(S(L[1]) \otimes V, L[1] \otimes V)$ into a DGLA. The space $\text{Hom}(S(L[1]), V)$ is then an abelian Lie subalgebra that is invariant under $[\lambda + \rho, -]$. Explicitly, we have for $f \in \text{Hom}(S(L[1]), V)$ homogeneous

\[
[\lambda + \rho, f] = \rho(\text{id}_S \otimes f)\Delta_S - (-1)^{|f|}f \circ d.
\]

Definition 61. The map $\delta := [\lambda + \rho, -]: \text{Hom}(S(L[1]), V) \rightarrow \text{Hom}(S(L[1]), V)$ is called the $L_\infty$-coboundary operator. The cohomology of the cochain complex $(\text{Hom}(S(L[1]), V), \delta)$ is called the $L_\infty$-algebra cohomology with values in $V$.

Remark 62. For $L$ and $V$ as in Section 4.3, we clearly have $\delta = D_{CE}$. If $L = \mathfrak{g}$ and $V$ are concentrated in degree zero, the décalage isomorphism (5) implies that

\[
\text{Hom}_p(S(\mathfrak{g}[1]), V) \cong \prod_{n \geq 1} \text{Hom}_{p-n}(\bigwedge^n \mathfrak{g}, V)
\]

\[
\cong \text{Hom}(\bigwedge^p \mathfrak{g}, V)
\]

for all $p \geq 1$. This way, we recover the usual Lie algebra cohomology.

Example 63 (The adjoint representation). The adjoint representation of $L$ on $L[1]$ is given by $S(L[1]) \otimes L[1] \rightarrow L[1], x \otimes y \mapsto \lambda(x \lor y)$. While there are now two distinct copies of $L[1]$ involved, it is evident by (43) that $\delta = [\lambda, -]$, the bracket being the one on $\text{Hom}(S(L[1]), L[1])$. This is the case discussed in [4].

5.2 $L_\infty$-structures induced by 2-cocycles

The description of $L_\infty$-structures, representations (up to homotopy) and the $L_\infty$-coboundary operator all by the same Lie bracket yields the following generalisation of Theorem 32.

Theorem 64. Let $L$ and $V$ be graded vector spaces and $\lambda \in \text{Hom}(S(L[1]), L[1]), \rho \in \text{Hom}(S(L[1]) \otimes V, V)$ and $\omega \in \text{Hom}(S(L[1]), V)$ be of degree one. Then $L[1] \oplus V, \lambda + \rho + \omega$ is an $L_\infty$-algebra if and only if $L[1], \lambda$ is an $L_\infty$-algebra, $\rho$ is a representation of $L$ on $V$ and $\omega$ is a $V$-valued cocycle.

Proof. The map $\frac{1}{2}[\lambda + \rho + \omega, \lambda + \rho + \omega] = \frac{1}{2}[\lambda, \lambda] + [\lambda, \rho] + \frac{1}{2}[\rho, \rho] + [\lambda + \rho, \omega]$ decomposes itself into linear maps

\[
\frac{1}{2}[\lambda, \lambda]: S(L[1]) \rightarrow L[1],
\]

\[
[\lambda, \rho] + \frac{1}{2}[\rho, \rho]: S(L[1]) \otimes V \rightarrow V,
\]

\[
[\lambda + \rho, \omega]: S(L[1]) \rightarrow V.
\]

The assertion then follows from Remark 55, Corollary 58 and the definition of $\delta$.

In terms of antisymmetric brackets, Theorem 64 characterises $L_\infty$-structures on $L \oplus V$ in which for each $n \in \mathbb{N}$, the $n$-ary bracket decomposes into linear maps

\[
\bigwedge^n L \rightarrow L,
\]

\[
\bigwedge^{n-1} L \otimes V \rightarrow V,
\]

\[
\bigwedge^n L \rightarrow V.
\]

These then correspond to cocycles in $\text{Hom}_1(S(L[1]), V[1]) \cong \text{Hom}_2(S(L[1]), V)$. So, it is the 2-cocycles that characterise these $L_\infty$-structures, as in the Lie algebra case (cf. [9], Proposition 7.5.18, p. 202).
\[ [\lambda_2, \lambda_m](x \vee y) = (\lambda_2 \circ \mu_S(\lambda_m \otimes \text{id}_S)\Delta_S)(x \vee y) + (-1)^{|x|}\lambda_m(x \vee d_2(y)) \]
\[ = \lambda_2 \circ \mu_S(\delta(x) \otimes \text{id}_S)\Delta_S(y) + (-1)^{|x|}\delta(x)(d_2(y)) \]
\[ = [\lambda_2, \delta(x)](y), \]
\[ \frac{1}{2}[\lambda_m, \lambda_m](x \vee y) = \sum (-1)^{|x|+|y|}\lambda_m(\lambda_m(x(1) \vee y(1)) \vee x(2) \vee y(2)) \]
\[ = \sum (-1)^{|x|+|y|}\delta(x(1))\delta(x(2))\delta(y(1)) \vee y(2)) \]
\[ = \frac{1}{2}\sum (-1)^{|x|}\delta(x(1))\delta(x(2))\delta(y(1)) \vee y(2)) \]
\[ + (-1)^{|x|}\delta(x(1))\delta(x(2))\delta(y(1)) \vee y(2)) \]
\[ = \left( \frac{1}{2}[\cdot, \cdot] \circ (\delta \otimes \delta)\Delta_S \right)(x(y), \)

5.3 Extensions of $L_\infty$-algebras

We conclude with a brief discussion of extensions of $L_\infty$-algebras. This puts some constructions we discussed in context. The notions are completely analogous to the Lie algebra case, see for example ([9], Sections 5.1.3 and 7.5.2).

A graded subspace $I \subset L$ of an $L_\infty$-algebra $(L[1], \lambda)$ is called an ideal if $\lambda(x \times y) \in I[1]$ for all $x \in I[1]$ and $y \in S(L[1])$. Then $L/I$ carries a canonical $L_\infty$-structure such that the projection $L \rightarrow L/I$ is a strict homomorphism of $L_\infty$-algebras. An ideal $I \subset L$ is always an $L_\infty$-subalgebra as in particular $\lambda(x) \in I[1]$ for all $x \in S(I[1])$.

**Definition 65.** An extension of an $L_\infty$-algebra $(L[1], \lambda_1)$ by another $L_\infty$-algebra $(L_2[1], \lambda_2)$ is an exact sequence of $L_\infty$-algebras and strict homomorphisms

\[ 0 \rightarrow L_2 \rightarrow L \rightarrow L_1 \rightarrow 0 \quad (44) \]

Given such an exact sequence (44), the graded subspace $L_2 \cong \ker(p) \subset L$ is an ideal and $p$ induces a strict isomorphism $L/L_2 \cong L_1$ of $L_\infty$-algebras.

We then always have $L \cong L_1 \oplus L_2$ (non-canonically) as graded vector spaces, so we are essentially concerned with $L_\infty$-structures on $L_1 \oplus L_2$ such that the canonical maps $L_2 \rightarrow L_1 \oplus L_2$ and $L_1 \oplus L_2 \rightarrow L_1$ are strict $L_\infty$-algebra homomorphisms. With the decomposition (40), we can decompose such an $L_\infty$-structure $\lambda : S((L_1 \oplus L_2)[1]) \rightarrow (L_1 \oplus L_2)[1]$ into linear degree one maps

\[ \lambda_1 : S(L_1[1]) \rightarrow L_1, \quad \omega : S(L_1[1]) \rightarrow L_2[1], \]
\[ \omega : S(L_2[1]) \rightarrow L_1[1], \quad \lambda_2 : S(L_2[1]) \rightarrow L_2[1], \]
\[ \omega : S(L_1[1]) \otimes S(L_2[1]) \rightarrow L_1[1], \quad \lambda_m : S(L_1[1]) \otimes S(L_2[1]) \rightarrow L_2[1]. \]

5.3.1 Abelian and central extensions

An $L_\infty$-algebra $L$ is called abelian if only its 1-ary bracket is nontrivial. An abelian $L_\infty$-algebra is then nothing else than a DG vector space.

An $L_\infty$-algebra extension $L_2 \rightarrow L \rightarrow L_1$ is called abelian if $L_2$ is abelian. The $L_\infty$-structures constructed in Theorem 64 are examples of abelian extensions of $L$ by $V$.

Similarly, an extension $L_2 \rightarrow L \rightarrow L_1$ is called central if $\lambda(x \vee y) = 0$ for $x \in L_2[1], y \in S(L[1])$. It is immediate that this is the case if and only if $L_2$ is abelian and $\lambda_m = 0$.

For abelian $L_2$, the central extensions $L_2 \rightarrow L_1 \oplus L_2 \rightarrow L_1$ are by Theorem 64 characterised by 2-cocycles of $L_1$ with values in the trivial representation of $L_1$ on $L_2$.

5.3.2 Semidirect sums

An $L_\infty$-algebra $((L_1 \oplus L_2)[1], \lambda)$ is said to be a semidirect sum of the $L_\infty$-algebras $(L_1[1], \lambda_1)$ and $(L_2[1], \lambda_2)$ if the canonical sequence $L_2 \rightarrow L_1 \oplus L_2 \rightarrow L_1$ is an $L_\infty$-algebra extension and if the canonical map $L_1 \rightarrow L_1 \oplus L_2$ is a strict homomorphism of $L_\infty$-algebras. This is clearly the case if and only if $\omega = 0$ in the decomposition above. A semidirect sum of $L_1$ and $L_2$ is therefore characterised by $\lambda_m$. Note that $L_1 \subset L_1 \oplus L_2$ is an ideal if and only if $\lambda_m = 0$. In this case, $L_1 \oplus L_2$ carries the $L_\infty$-structure $\lambda_1 + \lambda_2$ and is called the direct sum of $L_1$ and $L_2$.

For an arbitrary $\lambda_m \in \text{Hom}_S(S(L_1[1]) \otimes S(L_2[1]), L_2[1])$, the condition for $\lambda_1 + \lambda_2 + \lambda_m$ to define an $L_\infty$-structure on $L_1 \oplus L_2$ becomes

\[ [\lambda_1 + \lambda_2, \lambda_m] + \frac{1}{2}[\lambda_m, \lambda_m] = 0. \quad (45) \]

The isomorphism $\text{Hom}(S(L_1[1]) \otimes S(L_2[1]), L_2[1]) \cong \text{Hom}(S(L_1[1]), \text{Hom}(S(L_2[1]), L_2[1]))$ allows for the following characterisation of semidirect sums.

**Theorem 66.** Let $\lambda_m \in \text{Hom}(S(L_1[1]) \otimes S(L_2[1]), L_2[1])$ be of degree one. Then $\lambda_m$ satisfies (45) if and only if the corresponding linear degree one map $\delta : S(L_1[1]) \rightarrow \text{Hom}(S(L_2[1]), L_2[1])$ is a weak homomorphism of $L_\infty$-algebras in the sense that it satisfies (35).

**Proof.** Note that $\text{Hom}(S(L_1[1]) \otimes S(L_2[1]), L_2[1])$ is closed under $[\cdot, \cdot]$ and $[\lambda_1 + \lambda_2, \cdot]$. Therefore, (45) has only to be checked on $S(L_1[1]) \otimes S(L_2[1])$. Let $d_1$ and $d_2$ denote the codifferentials on $S(L_1[1])$ and $S(L_2[1])$, respectively. For $x \in S(L_1[1])$ and $y \in S(L_2[1])$, we then compute

\[ \text{See this equation above} \]

and $[\lambda_1, \lambda_m](x \vee y) = \lambda_m(d_1(x) \vee y) = (\delta \circ d_1)(x)(y)$.

**Example 67.** The $L_\infty$-structure on $L \oplus V$ induced by a representation of $L$ on $V$ is a semidirect sum. For
compliance with Theorem 66, note that $\mathfrak{gl}(L_2[1]) \subset \text{Hom}(\mathcal{S}(L_2[1]), L_2[1])$ is a Lie subalgebra.

References

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