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L_∞-algebras and their cohomology

Ben Reinhold*

University of Göttingen Mathematical Institute, 37073 Göttingen, Germany

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Abstract. We give an overview of different characterisations of L_∞-structures in terms of symmetric brackets and (co)differentials on the symmetric (co)algebra. We then do the same for their representations (up to homotopy) and approach L_∞-algebra cohomology using the commutator bracket on the space of coderivations of the symmetric coalgebra. This leads to abelian extensions of L_∞-algebras by 2-cocycles.

Keywords: L_∞-algebras / representations (up to homotopy) / L_∞-algebra cohomology / abelian extensions by 2-cocycles

1 Introduction

L_∞-algebras (also called *strongly homotopy Lie algebras*) were first introduced in [1] and [2] and are a generalisation of graded Lie algebras in which a system of antisymmetric n-ary brackets satisfies a generalised Jacobi identity. The first part of this article serves as a self-contained introduction to L_∞-algebras, in which we discuss different characterisations of L_∞-algebras and their representations (up to homotopy), closely following [3].

The L_∞-algebra cohomology with values in the adjoint representation was introduced in [4] using a Lie bracket on the space of cochains. We extend this approach to arbitrary representations, which leads to a characterisation of certain L_∞-algebras as abelian extensions of L_∞-algebras by 2-cocycles. This generalises a theorem from [5] that characterises certain L_∞-algebras in terms of Lie algebra cohomology.

This article is largely based on my same-titled Bachelor's thesis, which I wrote under the supervision of Chenchang Zhu at the University of Göttingen in 2018.

2 Mathematical background

In this section, we discuss exterior and symmetric powers, algebras and coalgebras in the graded framework. In particular, we show that antisymmetric and symmetric maps are related by a shift in degree and that coderivations of the symmetric coalgebra are in one-to-one correspondence with their weight one components. These results are later key to the characterisations of L_∞-structures in terms of symmetric brackets and codifferentials. The main references for this section are [3,4,6,7].

2.1 Graded vector spaces

A *graded vector space* is a vector space V together with a decomposition $V \cong \bigoplus_{p\in\mathbb{Z}} V_p$ for a family of vector spaces $\{V_p\}_{p\in\mathbb{Z}}$. An element $v \in V_p$ is then called *homogeneous of degree* p and we write $|v| = p$.

Here and subsequently, we assume all vector spaces to be over a fixed ground field $\Bbbk$ of characteristic zero. We always denote by V and W graded vector spaces and by $v_1, \ldots, v_n \in V$ arbitrary homogeneous elements.

A linear map $f\colon V \to W$ is called *homogeneous (of degree p)* if there is $p \in \mathbb{Z}$ such that $f(V_n) \subset W_{n+p}$ for all $n \in \mathbb{Z}$. We denote by $\mathrm{Hom}_p(V, W)$ the vector space of all homogeneous linear maps $V \to W$ of degree p and by $\mathrm{Hom}(V, W)$ the graded vector space $\bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_p(V, W)$. Elements in $\mathrm{Hom}_0(V, W)$ are also called *degree preserving*.

Note that we can identify ungraded vector spaces with graded ones that are *concentrated in degree zero*, that is $V_k = 0$ for $k \neq 0$.

There is a canonical grading on the direct sum of V and W given by $(V \oplus W)_p = V_p \oplus W_p$. The isomorphism

$$V \otimes W \cong \bigoplus_{p\in\mathbb{Z}} \bigoplus_{i+j=p} (V_i \otimes W_j)$$

allows us to define a grading on $V \otimes W$ by

$$(V \otimes W)_p = \bigoplus_{i+j=p} (V_i \otimes W_j)\,.$$

* e-mail: ben.reinhold@stud.uni-goettingen.de

This extends to a grading on $V^{\otimes n} := \bigotimes_{i=1}^n V$ given by

$$(V^{\otimes n})_p = \bigoplus_{i_1+\ldots+i_n=p} V_{i_1} \otimes \ldots \otimes V_{i_n}.$$

We denote by $\tau_{V,W}$ the linear degree preserving map

$$\tau_{V,W}\colon V \otimes W \to W \otimes V, \quad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v.$$

If $f \in \mathrm{Hom}(V, W)$ and $g \in \mathrm{Hom}(V', W')$ are homogeneous for graded vector spaces V' and W', we define the linear map $f \otimes g\colon V \otimes V' \to W \otimes W'$ by

$$(f \otimes g)(v \otimes v') = (-1)^{|v||g|} f(v) \otimes g(v') \qquad (1)$$

for $v \in V$, $v' \in V'$ homogeneous. Note that $|f \otimes g| = |f| + |g|$. This generalises to tensor products of three or more vector spaces in the obvious way and we abbreviate $f \otimes \ldots \otimes f\colon V^{\otimes n} \to W^{\otimes n}$ to $f^{\otimes n}$.

For the composition of such functions, (1) implies

$$(f' \otimes g') \circ (f \otimes g) = (-1)^{|g'||f|} (f' \circ f) \otimes (g' \circ g), \qquad (2)$$

when f' and g' are homogeneous linear maps with domains W and W', respectively.

When working in the framework of graded vector spaces, the general rule of thumb for the signs is that whenever two "graded symbols" of degree p and q, respectively, change their order in an equation, there should be the sign $(-1)^{|p||q|}$. This is called the *Koszul sign convention.*

We denote by $\mathfrak{S}_n$ the *symmetric group*, the group of all permutations of the set $\{1, \ldots, n\}$, and by $s_i \in \mathfrak{S}_n$ for $1 \leq i \leq n-1$ the transposition with $s_i(i) = i+1$ and $s_i(i+1) = i$. There are two canonical linear right actions of $\mathfrak{S}_n$ on $V^{\otimes n}$. These are given on the generating subset $\{s_1, \ldots s_{n-1}\} \subset \mathfrak{S}_n$ by

$$\begin{aligned}\hat{\varepsilon}(s_i)(v_1 \otimes \ldots \otimes v_n) &= (-1)^{|v_i||v_{i+1}|} v_1 \otimes \ldots \otimes v_{i-1} \\ &\quad \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \otimes \ldots \otimes v_n, \\ \hat{\chi}(s_i)(v_1 \otimes \ldots \otimes v_n) &= -(-1)^{|v_i||v_{i+1}|} v_1 \otimes \ldots \otimes v_{i-1} \\ &\quad \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \otimes \ldots \otimes v_n.\end{aligned}$$

We call $\hat{\varepsilon}$ and $\hat{\chi}$ the *(graded) symmetric* and *(graded) antisymmetric* action of $\mathfrak{S}_n$ on $V^{\otimes n}$, respectively. Note that $\hat{\varepsilon}(\sigma)$ is degree preserving for all $\sigma \in \mathfrak{S}_n$ as

$$\hat{\varepsilon}(\sigma)(v_1 \otimes \ldots \otimes v_n) = \pm v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}. \qquad (3)$$

We then denote the sign in (3) by $\varepsilon(\sigma; v_1, \ldots, v_n)$ and similarly by $\chi(\sigma; v_1, \ldots, v_n)$ the sign such that

$$\hat{\chi}(\sigma)(v_1 \otimes \ldots \otimes v_n) = \chi(\sigma; v_1, \ldots, v_n) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}.$$

We abbreviate $\varepsilon(\sigma; v_1, \ldots, v_n)$ and $\chi(\sigma; v_1, \ldots, v_n)$ to $\varepsilon(\sigma)$ and $\chi(\sigma)$, when no confusion can arise.

Let $U_S \subset V^{\otimes n}$ be the graded subspace spanned by all elements of the form

$$v_1 \otimes \ldots \otimes v_n - \hat{\varepsilon}(\sigma)(v_1 \otimes \ldots \otimes v_n)$$

for $\sigma \in \mathfrak{S}_n$. The space $\mathcal{S}^n(V) := V^{\otimes n}/U_S$ is called the nth symmetric power of V. Similarly, the nth *exterior power of* V is defined as the quotient of $V^{\otimes n}$ by the graded subspace spanned by all elements of the form

$$v_1 \otimes \ldots \otimes v_n - \hat{\chi}(\sigma)(v_1 \otimes \ldots \otimes v_n)$$

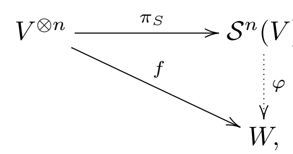
for $\sigma \in \mathfrak{S}_n$ and is denoted by $\bigwedge^n V$.

An n-linear map $f\colon V^n \to W$ is called *(graded) symmetric* if for all $\sigma \in \mathfrak{S}_n$

$$f(v_1, \ldots, v_n) = \varepsilon(\sigma) f(v_{\sigma(1)}, \ldots, v_{(n)})$$

holds. We can write this conveniently as $f \circ \hat{\varepsilon}(\sigma) = f$. Similarly, f is called *(graded) antisymmetric* if $f \circ \hat{\chi}(\sigma) = f$ for all $\sigma \in \mathfrak{S}_n$.

Proposition 1. *Let $f\colon V^{\otimes n} \to W$ be a symmetric linear map. There is a unique linear map $\varphi\colon \mathcal{S}^n(V) \to W$ such that the following diagram commutes:*



where $\pi_S\colon V^{\otimes n} \to \mathcal{S}^n(V)$ is the canonical projection.

Proof. As f is symmetric, it vanishes on the generators of U_S and factors through π_S to a linear map $\varphi\colon \mathcal{S}^n(V) \to W$ such that the diagram above commutes. This map is unique as π_S is surjective.

Remark 2. As the symmetric $\mathfrak{S}_n$-action on $V^{\otimes n}$ is degree preserving, $\mathcal{S}^n(V)$ inherits a canonical grading from $V^{\otimes n}$ such that π_S is degree preserving. It is then immediate that if f is homogeneous in Proposition 1, so is the map φ and $|\varphi| = |f|$. As π_S is symmetric by construction of $\mathcal{S}^n(V)$, Proposition 1 yields an isomorphism between the subspace of $\mathrm{Hom}(V^{\otimes n}, W)$ consisting of all symmetric maps and $\mathrm{Hom}(\mathcal{S}^n(V), W)$. An analogue of Proposition 1 holds for $\bigwedge^n V$ and induces an isomorphism between the subspace $\mathrm{Hom}(V^{\otimes n}, W)$ of all anti-symmetric maps and $\mathrm{Hom}(\bigwedge^n V, W)$.

An element in $V^{\otimes n}$ is called *symmetric* if it is invariant under the symmetric $\mathfrak{S}_n$-action on $V^{\otimes n}$. We claim that $\mathcal{S}^n(V)$ is isomorphic to the subspace of $V^{\otimes n}$ of all symmetric elements. Indeed, letting $v_1 \vee \ldots \vee v_n$ denote the image of $v_1 \otimes \ldots \otimes v_n$ under π_S, the linear map

$$\begin{aligned}&\varphi\colon \mathcal{S}^n(V) \to V^{\otimes n}, \\ &x_1 \vee \ldots \vee x_n \mapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}\end{aligned}$$

is well-defined and satisfies $\pi_S \circ \varphi = \mathrm{id}_{\mathcal{S}^n(V)}$ and $\varphi \circ \pi_S = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \hat{\varepsilon}(\sigma)$. As the latter is a projection of $V^{\otimes n}$ onto said subspace, the claim follows. A similar statement clearly holds for $\bigwedge^n V$.

For $n \in \mathbb{Z}$, we define the graded vector space $V[n]$ to be the vector space V with the grading defined by $V[n]_p = V_{p+n}$. We denote by $\downarrow^n\colon V \to V[n]$ the identity map on V, which becomes a linear isomorphism of degree $-n$, and by $\uparrow^n$ its inverse. We abbreviate $\downarrow^1$ and $\uparrow^1$

to $\downarrow$ and $\uparrow$, respectively. Note that $(\downarrow^{\otimes k})^{-1} = (-1)^{\frac{k(k-1)}{2}} \uparrow^{\otimes k}$ as a consequence of (2).

Proposition 3 (The décalage isomorphism)**.** *For $\sigma \in \mathfrak{S}_n$,*

$$\hat{\varepsilon}(\sigma) \circ \downarrow^{\otimes n} = \downarrow^{\otimes n} \circ \hat{\chi}(\sigma). \tag{4}$$

There is then a degree preserving isomorphism

$$\mathcal{S}^n(V[1]) \cong \left(\bigwedge\nolimits^n V\right)[n]. \tag{5}$$

Proof. Note that for the first part, we only have to check (4) on the generating subset $\{s_1, \ldots, s_{n-1}\} \subset \mathfrak{S}_n$. This is an easy computation left to the reader. Let $\pi_A \colon V^{\otimes n} \to \bigwedge^n V$ be the canonical projection. The linear maps

$$\pi_S \circ \downarrow^{\otimes n} \colon V^{\otimes n} \to \mathcal{S}(V[1]),$$
$$(-1)^{\frac{n(n-1)}{2}} \pi_A \circ \uparrow^{\otimes n} \colon (V[1])^{\otimes n} \to \bigwedge\nolimits^n V$$

are then antisymmetric and symmetric, respectively. The induced linear maps between $\mathcal{S}(V[1])$ and $\bigwedge^n V$ are then easily seen to be inverse to each other. As these maps are of degree $-n$ and n, respectively, we obtain a degree preserving isomorphism $\mathcal{S}(V[1]) \cong \left(\bigwedge^n V\right)[n]$.

Corollary 4. *There is for each $p \in \mathbb{Z}$ a one-to-one correspondence between symmetric linear maps $\lambda \colon (V[1])^{\otimes n} \to V[1]$ of degree p and antisymmetric linear maps $l \colon V^{\otimes n} \to V$ of degree $p + 1 - n$ given by*

$$l = \uparrow \circ \lambda \circ \downarrow^{\otimes n},$$
$$\lambda = (-1)^{\frac{n(n-1)}{2}} \downarrow \circ\, l \circ \uparrow^{\otimes n}.$$

A *differential* on the graded vector space V is a linear map $d \colon V \to V$ of degree one such that $d^2 = 0$. We then call the pair (V, d) a *differential graded vector space* (*DG vector space* for short). A *homomorphism between DG vector spaces* (V, d) and (W, d') is a degree preserving linear map $f \colon V \to W$ such that $d' \circ f = f \circ d$.

DG vector spaces are sometimes called *cochain complexes*. Given a cochain complex (V, d), one then calls an element $v \in V_n$ an *n-cocycle* if $d(v) = 0$ and an *n-coboundary* if $v = d(w)$ for some $w \in V_{n-1}$. The graded vector space $H(V) = \ker(d)/\operatorname{im}(d)$ measures the non-exactness of the sequence

$$\cdots \xrightarrow{d} V_{n-1} \xrightarrow{d} V_n \xrightarrow{d} V_{n+1} \xrightarrow{d} \cdots$$

and is called the *cohomology* of (V, d). We then call $H_n(V) = \frac{n\text{-cocycles}}{n\text{-coboundaries}}$ the nth *cohomology group*.

2.2 Graded algebras

By an *algebra* we mean a vector space A together with a linear map $\mu \colon A \otimes A \to A$; the multiplication μ is in general not assumed to be associative.

A *graded algebra* A is an algebra that is also a graded vector space in which the multiplication is degree preserving. If also $ab = (-1)^{|a||b|} ba$ for all $a, b \in A$, we call A *(graded) commutative*. A *homomorphism of graded algebras* is a degree preserving algebra homomorphism.

A (two-sided) ideal I in A is called *homogeneous* if $I \subset A$ is a graded subspace. Note that an ideal is homogeneous if and only if it is spanned by homogeneous elements.

Remark 5. If $I \subset A$ is a homogeneous ideal, the canonical isomorphism $A/I \cong \bigoplus_{n \in \mathbb{Z}} A_n/I_n$ makes A/I into a graded algebra such that the canonical projection $A \to A/I$ is a homomorphism of graded algebras.

Remark 6. Let A and B be two graded associative algebras. The multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|} aa' \otimes bb'. \tag{6}$$

for $a, a' \in A$, $b, b' \in B$ homogeneous makes $A \otimes B$ into a graded associative algebra. If A and B are both unital/commutative, then so is $A \otimes B$.

Example 7 (The tensor algebra)**.** We denote by $T(V) := \bigoplus_{n \geq 0} V^{\otimes n}$ the *tensor algebra* of V. It carries the multiplication induced by the canonical isomorphism $V^{\otimes r} \otimes V^{\otimes s} \cong V^{\otimes (r+s)}$, making it into a unital associative algebra. The grading on $T(V)$ induced by the grading on $V^{\otimes n}$ is given by

$$T(V)_p = \bigoplus_{i_1 + \ldots + i_n = p} V_{i_1} \otimes \ldots \otimes V_{i_n}$$

and is called the *interior grading*. On the other hand, $T(V)$ carries the grading given by $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$, which is called the *exterior grading* or *grading by weight*. If not specified otherwise, we always understand $T(V)$ to carry its interior grading. Note that both gradings make $T(V)$ into a graded algebra.

Example 8 (The symmetric and exterior algebra)**.** Let $I_S \subset T(V)$ be the two-sided homogeneous ideal generated by elements of the form $v_1 \otimes v_2 - (-1)^{|v_1||v_2|} v_2 \otimes v_1$. We call $\mathcal{S}(V) := T(V)/I_S$ the *symmetric algebra* of V. Similarly, the *exterior algebra* of V, denoted by $\bigwedge V$, is defined as the quotient of $T(V)$ by the two-sided homogeneous ideal generated by elements of the form $v_1 \otimes v_2 + (-1)^{|v_1||v_2|} v_2 \otimes v_1$. We denote the multiplication in $\mathcal{S}(V)$ and $\bigwedge V$ by $\vee$ and $\wedge$, respectively. Note that $\mathcal{S}(V)$ and $\bigwedge V$ also admit an *exterior grading* or *grading by weight*: as $V^{\otimes n} \cap I_S = U_S$, we have $\mathcal{S}(V) = \bigoplus_{n \geq 0} \mathcal{S}^n(V)$ and similarly $\bigwedge V = \bigoplus_{n \geq 0} \bigwedge^n V$.

It is easy to see that if A is a graded unital associative algebra, there is for each linear degree preserving map $f \colon V \to A$ a unique homomorphism of unital graded algebras $\varphi \colon T(V) \to A$ that agrees on V with f (see for example [6], Proposition 1.1.1). It is then immediate that if A is commutative, φ factors to a unique homomorphism of unital graded algebras $\mathcal{S}(V) \to A$. Applying this to the linear map

$$V \oplus W \to \mathcal{S}(V) \otimes \mathcal{S}(W), \quad (v, w) \mapsto v \otimes 1 + 1 \otimes w$$

yields a homomorphism of graded unital algebras $\mathcal{S}(V \oplus W) \to \mathcal{S}(V) \otimes \mathcal{S}(W)$ that is easily seen to be an isomorphism with inverse $\mathcal{S}(V) \otimes \mathcal{S}(W) \to \mathcal{S}(V \oplus W)$, $v \otimes w \mapsto v \vee w$. With a slight modification of the sign in

(6), similar arguments show that $\bigwedge(V \oplus W) \cong \bigwedge V \otimes \bigwedge W$. In particular, we have

$$\mathcal{S}^n(V \oplus W) \cong \bigoplus_{p+q=n} \mathcal{S}^p(V) \otimes \mathcal{S}^q(W), \tag{7}$$

$$\bigwedge\nolimits^n(V \oplus W) \cong \bigoplus_{p+q=n} \bigwedge\nolimits^p V \otimes \bigwedge\nolimits^q W. \tag{8}$$

For a graded algebra A, a *derivation of A of degree p* is a linear map $d\colon A \to A$ of degree p satisfying

$$d(ab) = d(a)b + (-1)^{p|a|} a d(b)$$

for all $a, b \in A$ homogeneous. We denote by $\mathrm{Der}_p(A)$ the vector space of all derivations of A of degree p and by $\mathrm{Der}(A)$ the graded vector space $\bigoplus_{p\in\mathbb{Z}} \mathrm{Der}_p(A)$. A *differential on the graded algebra A* is an element $d \in \mathrm{Der}(A)$ of degree one such that $d^2 = 0$. The pair (A, d) is then called a *differential graded algebra* (*DG algebra* for short). A *homomorphism of DG algebras* is a homomorphism of graded algebras that is also a homomorphism of DG vector spaces.

2.3 Graded Lie algebras and unshuffle permutations

Definition 9. A *graded Lie algebra* is a graded vector space L together with a (graded) antisymmetric degree preserving linear map $[\cdot,\cdot]\colon L \otimes L \to L$ called the *Lie bracket* satisfying the *(graded) Jacobi identity*

$$[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]] = 0 \tag{9}$$

for all $x, y, z \in L$ homogeneous.

If L is ungraded, we recover the usual definition of a Lie algebra. Note that (9) is nothing else than $[x, \cdot\,]$ being a derivation of the graded algebra $(L, [\cdot,\cdot])$.

Example 10. For a graded associative algebra A, we define the *graded commutator* $[\cdot,\cdot]\colon A \otimes A \to A$ by $[a,b] = ab - (-1)^{|a||b|}ba$ for $a, b \in A$ homogeneous. This makes A into a graded Lie algebra. In particular, $\mathfrak{gl}(V) := \mathrm{Hom}(V,V)$ becomes a graded Lie algebra. If V is itself a graded algebra, one can check that $\mathrm{Der}(V) \subset \mathfrak{gl}(V)$ is a Lie subalgebra.

Definition 11. A *differential graded Lie algebra* (*DGLA* for short) is a DG algebra in which the underlying algebra is a graded Lie algebra.

Example 12. Let L be a graded Lie algebra and $x \in L$ a degree one element such that $\frac{1}{2}[x,x] = 0$. Then $d := [x, \cdot\,]$ satisfies $d^2 = 0$ by the Jacobi identity (9) and $(L, [\cdot,\cdot], d)$ is a DGLA. In particular, for (V, ∂) a DG vector space, this makes $(\mathfrak{gl}(V), [\cdot,\cdot], [\partial, \cdot\,])$ canonically into a DGLA as $\frac{1}{2}[\partial,\partial] = \partial^2 = 0$.

Definition 13. For a DGLA $(L, [\cdot,\cdot], d)$, a *Maurer–Cartan element* is an element $x \in L$ of degree one such that

$$d(x) + \frac{1}{2}[x,x] = 0. \tag{10}$$

The equation (10) is called the *Maurer–Cartan equation*.

Example 14. Let $(L, [\cdot,\cdot], d = [x, \cdot\,])$ be as in Example 12. For $y \in L$ of degree one, we then have $\frac{1}{2}[x+y, x+y] = 0$ if and only if y satisfies the Maurer–Cartan equation.

For $0 \le i \le n$, an $(i, n-i)$-*unshuffle* is a permutation $\sigma \in \mathfrak{S}_n$ satisfying $\sigma(1) < \ldots < \sigma(i)$ and $\sigma(i+1) < \ldots < \sigma(n)$. Following the notation in [3], we denote the set of all $(i, n-i)$-unshuffles by $\mathrm{Sh}^{-1}_{i,n-i} \subset \mathfrak{S}_n$. Using the antisymmetry of the Lie bracket, one can rewrite (9) as

$$\sum_{\sigma \in \mathrm{Sh}^{-1}_{2,1}} \chi(\sigma)[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}] = 0 \tag{11}$$

for all $x_1, x_2, x_3 \in L$ homogeneous.

Lemma 15. *Each element $\sigma \in \mathfrak{S}_n$ has for each $i \in \{0, \ldots, n\}$ a unique decomposition $\sigma = \tau(\alpha, \beta)$, where $\tau \in \mathrm{Sh}^{-1}_{i,n-i}$ and $(\alpha, \beta) \in \mathfrak{S}_i \times \mathfrak{S}_{n-i}$. Here, $\mathfrak{S}_i \times \mathfrak{S}_{n-i}$ is considered as a subgroup of $\mathfrak{S}_n$ in the obvious way.*

Proof. Clearly, τ has to be the unique $(i, n-i)$-unshuffle such that $\{\tau(1), \ldots, \tau(i)\} = \{\sigma(1), \ldots, \sigma(i)\}$ and $\{\tau(i+1), \ldots, \tau(n)\} = \{\sigma(i+1), \ldots, \sigma(n)\}$. We then have $\tau^{-1}\sigma \in \mathfrak{S}_i \times \mathfrak{S}_{n-i}$.

2.4 Graded coalgebras

A *(graded) coalgebra* (C, Δ) is a graded vector space C together with a degree preserving linear map $\Delta\colon C \to C \otimes C$ called the *coproduct*. If the diagram

$$\begin{array}{ccc} C & \xrightarrow{\ \Delta\ } & C \otimes C \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle \Delta \otimes \mathrm{id}_C} \\ C \otimes C & \xrightarrow[\mathrm{id}_C \otimes \Delta]{} & C \otimes C \otimes C \end{array} \tag{12}$$

commutes, C is called *coassociative*. We call C *counital* if there is a degree preserving linear map $\varepsilon\colon C \to \Bbbk$ such that the diagram

$$\begin{array}{ccccc} & & C & & \\ & \swarrow\!\!= & \Big\downarrow{\scriptstyle \Delta} & =\!\!\searrow & \\ \Bbbk \otimes C & \xleftarrow[\varepsilon \otimes \mathrm{id}_C]{} & C \otimes C & \xrightarrow[\mathrm{id}_C \otimes \varepsilon]{} & C \otimes \Bbbk \end{array} \tag{13}$$

commutes. The map ε is then called the *counit* of C. If $\tau_{C,C} \circ \Delta = \Delta$, then C is called *cocommutative*. A linear degree preserving map $f\colon C \to D$ between coalgebras (C, Δ_C) and (D, Δ_D) is called a *homomorphism of coalgebras* if

$$(f \otimes f)\Delta_C = \Delta_D \circ f. \tag{14}$$

If C and D are counital with counits ε and η, respectively, and if f also satisfies $\eta \circ f = \varepsilon$, we call f a *homomorphism of counital coalgebras.*

For a coassociative coalgebra (C,Δ) and $n\in\mathbb{N}$, we define the *iterated coproduct* $\Delta^n\colon C\to C^{\otimes(n+1)}$ by $\Delta^0=\mathrm{id}_C$ and $\Delta^n=(\Delta\otimes\mathrm{id}_C\otimes\ldots\otimes\mathrm{id}_C)\Delta^{n-1}$ for $n\geq 1$. It is convenient to then use *Sweedler notation* and to write $\Delta^n(x)\in C^{\otimes(n+1)}$ for $x\in C$ as

$$\Delta^n(x)=\sum x_{(1)}\otimes\ldots\otimes x_{(n+1)}.$$

In this notation, for example, the condition for C to be cocommutative becomes $\sum x_{(1)}\otimes x_{(2)}=\sum(-1)^{|x_{(1)}||x_{(2)}|}x_{(2)}\otimes x_{(1)}$ for all $x\in C$.

Lemma 16. *Let (C,Δ_C) be a coassociative coalgebra. Then for all $p,q\in\mathbb{N}$,*

$$(\Delta_C^p\otimes\Delta_C^q)\Delta_C=\Delta_C^{p+q+1}. \tag{15}$$

If (D,Δ_D) is another coassociative coalgebra and if $f\colon C\to D$ is a coalgebra homomorphism,

$$f^{\otimes(n+1)}\circ\Delta_C^n=\Delta_D^n\circ f \tag{16}$$

holds for all $n\in\mathbb{N}$.

Proof. One obtains (15) and (16) by iterating (12) and (14); the details are left to the reader or can be found in ([7], Lemma-Definition VIII.10).

2.4.1 Coaugmented coalgebras

A *coaugmented coalgebra* (C,Δ,ε,u) is a counital coassociative coalgebra (C,Δ,ε) together with a homomorphism of counital coalgebras $u\colon\Bbbk\to C$. The coproduct on $\Bbbk$ is given by $1_\Bbbk\mapsto 1_\Bbbk\otimes 1_\Bbbk$ and its counit is the identity on $\Bbbk$. Denoting $u(1_\Bbbk)\in C$ as 1, the conditions for u to be a homomorphism of counital coalgebras become $\Delta(1)=1\otimes 1$ and $\varepsilon\circ u=\mathrm{id}_\Bbbk$. A *homomorphism between coaugmented coalgebras* C and D is a homomorphism of counital coalgebras $f\colon C\to D$ such that $f(1)=1$.

Given a coaugmented coalgebra (C,Δ,ε,u), set $\overline{C}:=\ker(\varepsilon)$. We claim that $C\cong\overline{C}\oplus\Bbbk$. Indeed, as $\varepsilon\circ u=\mathrm{id}_\Bbbk$, the short exact sequence of graded vector spaces

$$0\to\ker(\varepsilon)\hookrightarrow C\xrightarrow{\varepsilon}\Bbbk\to 0$$

splits. For $x\in\overline{C}$, we then define $\overline{\Delta}(x):=\Delta(x)-x\otimes 1-1\otimes x$. Using (13) and $\varepsilon(x)=0$, one easily sees that

$$\overline{\Delta}(x)\in\ker(\varepsilon\otimes\mathrm{id}_C)\cap\ker(\mathrm{id}_C\otimes\varepsilon)=\overline{C}\otimes\overline{C};$$

for the last equality, note that $C\otimes C\cong(\overline{C}\otimes\overline{C})\oplus(\overline{C}\otimes\Bbbk)\oplus(\Bbbk\otimes\overline{C})\oplus(\Bbbk\otimes\Bbbk)$. We call $\overline{\Delta}\colon\overline{C}\to\overline{C}\otimes\overline{C}$ the *reduced coproduct* on $\overline{C}$. It is straightforward to check that $(\overline{C},\overline{\Delta})$ is a coassociative coalgebra.

Conversely, given a coassociative coalgebra $(\overline{C},\overline{\Delta})$, we define a coproduct on $C:=\overline{C}\oplus\Bbbk$ by $\Delta(1)=1\otimes 1$ and $\Delta(x)=\overline{\Delta}(x)+x\otimes 1+1\otimes x$ for $x\in\overline{C}$. This makes C into a coaugmented coalgebra; the counit and coaugmentation map are given by the projection $C\to\Bbbk$ and the inclusion $\Bbbk\hookrightarrow C$, respectively. These constructions are clearly inverse to each other (up to isomorphism).

Let C and D be coaugmented coalgebras with counits ε and η, respectively. A linear degree preserving map $f\colon C\to D$ satisfying $\eta\circ f=\varepsilon$ and $f(1)=1$ decomposes as

$$f=\overline{f}\oplus\mathrm{id}_\Bbbk\colon\overline{C}\oplus\Bbbk\to\overline{D}\oplus\Bbbk$$

for a unique degree preserving linear map $\overline{f}\colon\overline{C}\to\overline{D}$. It is then easy to see that f is a homomorphism of coaugmented coalgebras if and only if $\overline{f}$ is a homomorphism of coalgebras. This yields a one-to-one correspondence between coalgebra homomorphisms $\overline{C}\to\overline{D}$ and homomorphisms of coaugmented coalgebras $C\to D$.

Loosely speaking, this let us choose if we want to work with coaugmented coalgebras or non-coaugmented ones. In more technical terms, we have an equivalence between the category of coaugmented coalgebras and the category of coassociative coalgebras.

We call a coaugmented coalgebra (C,Δ) *conilpotent* if for all $x\in\overline{C}$ there is an $n\in\mathbb{N}$ such that $\overline{\Delta}^n(x)=0$.

2.4.2 Examples of coalgebras

There is a coproduct $\overline{\Delta}_A$ on $\overline{T}(V):=\bigoplus_{n\geq 1}V^{\otimes n}$ given by

$$\overline{\Delta}_A(v_1\ldots v_n)=\sum_{i=1}^{n-1}(v_1\ldots v_i)\otimes(v_{i+1}\ldots v_n),$$

where we now denote the multiplication in $T(V)$ by concatenation to avoid ambiguities. This makes $\overline{T}(V)$ into a coassociative graded coalgebra. The induced coaugmented coalgebra $(T(V),\Delta_A)$ is called the *tensor coalgebra*. Inductively, one finds

$$\overline{\Delta}_A^m(v_1\ldots v_n)=\sum_{1\leq i_1<\ldots<i_m<n}(v_1\ldots v_{i_1})\otimes\ldots\otimes(v_{i_m+1}\ldots v_n), \tag{17}$$

which shows that $T(V)$ is conilpotent.

Proposition 17. *Let (C,Δ) be a conilpotent coalgebra and $f\colon\overline{C}\to V$ a linear degree preserving map. There is a unique homomorphism of coaugmented coalgebras $\tilde{f}\colon C\to T(V)$ such that $f=\mathrm{pr}_V\circ\tilde{f}$, where here and subsequently, $\mathrm{pr}_{(\,\cdot\,)}$ denotes the projection onto a subspace under a given decomposition.*

Proof. It clearly suffices to show that there is a unique homomorphism of coalgebras $\tilde{f}\colon\overline{C}\to\overline{T}(V)$ satisfying $\mathrm{pr}_V\circ\tilde{f}=f$. For the uniqueness, assume that there is such $\tilde{f}$. By Lemma 16, we have

$$\tilde{f}^{\otimes(n+1)}\circ\overline{\Delta}^n=\overline{\Delta}_A^n\circ\tilde{f}$$

for all $n\in\mathbb{N}$. Composing both sides with $\mathrm{pr}_V^{\otimes(n+1)}$ and noting that $\mathrm{pr}_V^{\otimes(n+1)}\circ\overline{\Delta}_A^n=\mathrm{pr}_{V^{\otimes(n+1)}}$ then yields

$$\mathrm{pr}_{V^{\otimes(n+1)}}\circ\tilde{f}=\mathrm{pr}_V^{\otimes(n+1)}\circ\tilde{f}^{\otimes(n+1)}\circ\overline{\Delta}^n=f^{\otimes(n+1)}\circ\overline{\Delta}^n.$$

$$\begin{aligned}
&\overline{\Delta}_A(N(v_1 \vee \cdots \vee v_n))\\
&= \sum_{i=1}^{n-1} \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)(v_{\sigma(1)} \dots v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \dots v_{\sigma(n)})\\
&= \sum_{i=1}^{n-1} \sum_{\tau\in\mathrm{Sh}_{i,n-i}^{-1}} \sum_{(\alpha,\beta)\in\mathfrak{S}_i\times\mathfrak{S}_{n-i}} \varepsilon(\tau(\alpha,\beta))(v_{(\tau(\alpha,\beta))(1)} \dots v_{(\tau(\alpha,\beta))(i)}) \otimes (v_{(\tau(\alpha,\beta))(i+1)} \dots v_{(\tau(\alpha,\beta))(n)})\\
&= \sum_{i=1}^{n-1} \sum_{\tau\in\mathrm{Sh}_{i,n-i}^{-1}} \varepsilon(\tau) \sum_{(\alpha,\beta)\in\mathfrak{S}_i\times\mathfrak{S}_{n-i}} \hat{\varepsilon}(\alpha)(v_{\tau(1)} \dots v_{\tau(i)}) \otimes \hat{\varepsilon}(\beta)(v_{\tau(i+1)} \dots v_{\tau)(n)})\\
&= \sum_{i=1}^{n-1} \sum_{\tau\in\mathrm{Sh}_{i,n-i}^{-1}} \varepsilon(\tau) N(v_1 \vee \dots \vee v_i) \otimes N(v_{i+1} \vee \dots \vee v_n).
\end{aligned}$$

This shows that $\tilde{f}$ is completely determined by f and therefore unique. For the existence, consider the linear map $\sum_{n=0}^{\infty} \overline{\Delta}^n \colon \overline{C} \to \overline{T}(\overline{C})$. This is well-defined as C is conilpotent. A straightforward computation using Lemma 16 and $\overline{\Delta}_A\big|_V = 0$ shows that

$$\left(\sum_{n=0}^{\infty} \overline{\Delta}^n \otimes \sum_{n=0}^{\infty} \overline{\Delta}^n\right) \overline{\Delta} = \overline{\Delta}_A \circ \sum_{n=0}^{\infty} \overline{\Delta}^n,$$

so that $\sum_{n=0}^{\infty} \overline{\Delta}^n$ is a coalgebra homomorphism. As $\overline{T}(f) = \bigoplus_{n\geq 1} f^{\otimes n} \colon \overline{T}(\overline{C}) \to \overline{T}(V)$ is easily seen to be also a coalgebra homomorphism, $\tilde{f} := \overline{T}(f) \circ \sum_{n=0}^{\infty} \overline{\Delta}^n \colon C \to \overline{T}(V)$ is a homomorphism of coalgebras with $\mathrm{pr}_V \circ \tilde{f} = f$.

Let $\overline{\mathcal{S}}(V) := \bigoplus_{n\geq 1} \mathcal{S}^n(V)$. Consider the linear maps

$$\begin{aligned}
&\pi \colon \overline{T}(V) \to \overline{\mathcal{S}}(V), \quad v_1 \otimes \dots \otimes v_n \mapsto \frac{1}{n!} v_1 \vee \dots \vee v_n,\\
&N \colon \overline{\mathcal{S}}(V) \to \overline{T}(V),\\
&v_1 \vee \dots \vee v_n \mapsto \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma) v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}.
\end{aligned}$$

It is immediate that $\pi \circ N = \mathrm{id}_S$, where here and subsequently, we abbreviate $\overline{\mathcal{S}}(V)$ and $\mathcal{S}(V)$ to S in subscripts. Using Lemma 15, we compute

See this equation above.

This shows that $\mathrm{im}(N) \subset \overline{T}(V)$ is a subcoalgebra and induces a coproduct on $\overline{\mathcal{S}}(V) \cong \mathrm{im}(N)$. As a subcoalgebra of a coassociative coalgebra is clearly coassociative itself, $\mathcal{S}(V)$ becomes a coaugmented coalgebra with the coproduct $\Delta_S \colon \mathcal{S}(V) \to \mathcal{S}(V) \otimes \mathcal{S}(V)$ given by

$$\begin{aligned}
\Delta_S(v_1 \vee \dots \vee v_n) = \sum_{i=0}^{n} \sum_{\tau\in\mathrm{Sh}_{i,n-i}^{-1}} &\varepsilon(\tau)(v_{\tau(1)} \vee \dots \vee v_{\tau(i)})\\
&\otimes (v_{\tau(i+1)} \vee \dots \vee v_{\tau(n)}). \qquad (18)
\end{aligned}$$

It is immediate that $\mathcal{S}(V)$ is conilpotent, as it is a subcoalgebra of $T(V)$. We claim that $\mathcal{S}(V)$ is even cocommutative. Indeed, let $\sigma_i \in \mathfrak{S}_n$ be for $0 \leq i \leq n$ the permutation given by $(\sigma_i(1), \dots, \sigma_i(n)) = (i+1, \dots, n, 1, \dots, i)$. We then have $\mathrm{Sh}_{i,n-i}^{-1} \sigma_i = \mathrm{Sh}_{n-i,i}^{-1}$ and therefore

$$\begin{aligned}
&\tau_{S,S} \circ \Delta_S(v_1 \vee \dots \vee v_n)\\
&= \sum_{i=0}^{n} \sum_{\tau\in\mathrm{Sh}_{i,n-i}^{-1}} \varepsilon(\tau\sigma_i)(v_{(\tau\sigma_i)(1)} \vee \dots \vee v_{(\tau\sigma_i)(n-i)})\\
&\quad \otimes (v_{(\tau\sigma_i)(n-i+1)} \vee \dots \vee v_{(\tau\sigma_i)(n)})\\
&= \sum_{i=0}^{n} \sum_{\tau\in\mathrm{Sh}_{n-i,i}^{-1}} \varepsilon(\tau)(v_{\tau(1)} \vee \dots \vee v_{\tau(i)})\\
&\quad \otimes (v_{\tau(i+1)} \vee \dots \vee v_{\tau(n)})\\
&= \Delta_S(v_1 \vee \dots \vee v_n).
\end{aligned}$$

Proposition 18. *Let (C, Δ) be a cocommutative conilpotent coalgebra and $f \colon \overline{C} \to V$ a degree preserving linear map. There is a unique homomorphism of coaugmented coalgebras $\tilde{f} \colon C \to \mathcal{S}(V)$ such that $f = \mathrm{pr}_V \circ \tilde{f}$.*

Proof. As in the proof of Proposition 17, it suffices to show that there is a unique homomorphism of coalgebras $\tilde{f} \colon \overline{C} \to \overline{\mathcal{S}}(V)$ satisfying $\mathrm{pr}_V \circ \tilde{f} = f$. Recall that $\overline{T}(f) \circ \sum_n \overline{\Delta}^n$ is the unique coalgebra homomorphism $\overline{C} \to \overline{T}(V)$ extending f. For $0 \leq i \leq n-1$, we have

$$\begin{aligned}
&(\mathrm{id}_V^{\otimes i} \otimes \tau_{C,C} \otimes \mathrm{id}_V^{\otimes(n-i-1)}) \overline{T}(f) \circ \overline{\Delta}^n\\
&\quad = \overline{T}(f)(\mathrm{id}_C^{\otimes i} \otimes \tau_{C,C} \otimes \mathrm{id}_C^{\otimes(n-i-1)}) \overline{\Delta}^n\\
&\quad = \overline{T}(f)(\mathrm{id}_C^{\otimes i} \otimes (\tau_{C,C} \circ \overline{\Delta}) \otimes \mathrm{id}_C^{\otimes(n-i-1)}) \overline{\Delta}^{n-1}\\
&\quad = \overline{T}(f)(\mathrm{id}_C^{\otimes i} \otimes \overline{\Delta} \otimes \mathrm{id}_C^{\otimes(n-i-1)}) \overline{\Delta}^{n-1}\\
&\quad = \overline{T}(f) \circ \overline{\Delta}^n
\end{aligned}$$

since C is cocommutative. As $(\mathrm{id}_V^{\otimes i} \otimes \tau_{C,C} \otimes \mathrm{id}_V^{\otimes(n-i-1)}) = \hat{\varepsilon}(s_i)$, the image of $\overline{T}(f) \circ \sum_n \overline{\Delta}^n$ is contained in the subspace of $\overline{T}(V)$ of all symmetric elements, which is $\mathrm{im}(N)$.

We obtain an induced homomorphism of coalgebras

$$\begin{aligned} \tilde{f} = \pi \circ \overline{T}(f) \circ \sum_{n=0}^{\infty} \overline{\Delta}^n \colon \overline{C} \to \overline{\mathcal{S}}(V), \\ x \mapsto \frac{1}{n!} \sum_{n=1}^{\infty} f(x_{(1)}) \vee \ldots \vee f(x_{(n)}) \end{aligned} \tag{19}$$

with $\mathrm{pr}_V \circ \tilde{f} = f$. Similarly, a coalgebra homomorphism $\tilde{f} \colon \overline{C} \to \overline{\mathcal{S}}(V)$ gives rise to a coalgebra homomorphism $N \circ \tilde{f} \colon \overline{C} \to \overline{T}(V)$ that is uniquely determined by $\mathrm{pr}_V \circ N \circ \tilde{f} = \mathrm{pr}_V \circ \tilde{f}$ by Proposition 17. As N is injective, this shows uniqueness of $\tilde{f}$.

Example 19. A linear degree preserving map $f \colon V \to W$ can be extended by zero to a linear map $\overline{\mathcal{S}}(V) \to W$. The induced homomorphism of coaugmented coalgebras $\mathcal{S}(V) \to \mathcal{S}(W)$ is denoted by $\mathcal{S}(f)$ and is given by $\mathcal{S}(f)(v_1 \vee \ldots \vee v_n) = f(v_1) \vee \ldots \vee f(v_n)$.

2.4.3 Comodules and coderivations

Let (C, Δ) be a coassociative coalgebra. A *left comodule over* C is a graded vector space M together with a degree preserving linear map $\Delta_l \colon M \to C \otimes M$ satisfying

$$(\Delta \otimes \mathrm{id}_M)\Delta_l = (\mathrm{id}_C \otimes \Delta_l)\Delta_l. \tag{20}$$

Similarly, a *right comodule over* C is a graded vector space M together with a degree preserving linear map $\Delta_r \colon M \to M \otimes C$ such that

$$(\mathrm{id}_M \otimes \Delta)\Delta_r = (\Delta_r \otimes \mathrm{id}_C)\Delta_r. \tag{21}$$

If M is both a left and a right comodule over C and if the compatibility relation

$$(\Delta_l \otimes \mathrm{id}_C)\Delta_r = (\mathrm{id}_C \otimes \Delta_r)\Delta_l \tag{22}$$

is satisfied, M is called a *(bi)comodule over* C. Given such M, we define a *coderivation of degree* p to be a homogeneous linear map $d \colon M \to C$ of degree p such that

$$\Delta \circ d = (d \otimes \mathrm{id}_C)\Delta_r + (\mathrm{id}_C \otimes d)\Delta_l. \tag{23}$$

We denote the vector space of all these maps by $\mathrm{Coder}_p(M, C)$ and by $\mathrm{Coder}(M, C)$ the graded vector space $\bigoplus_{p \in \mathbb{Z}} \mathrm{Coder}_p(M, C)$.

Let (C, Δ_C) and (D, Δ_D) be coassociative coalgebras and $f \colon D \to C$ a coalgebra homomorphism. Then $\Delta_r := (\mathrm{id}_D \otimes f)\Delta_D$ and $\Delta_l := (f \otimes \mathrm{id}_D)\Delta_D$ make D into a comodule over C. In particular, C is a comodule over itself and we abbreviate $\mathrm{Coder}(C, C)$ to $\mathrm{Coder}(C)$. If C and D are coaugmented and if f is a homomorphism of coaugmented coalgebras, the comodule structure is *compatible with the counit* in the sense that the diagram

$$\begin{array}{ccccc} C \otimes D & \xleftarrow{\Delta_l} & D & \xrightarrow{\Delta_r} & D \otimes C \\ \downarrow{\scriptstyle \varepsilon \otimes \mathrm{id}_D} & & \| & & \downarrow{\scriptstyle \mathrm{id}_D \otimes \varepsilon} \\ \Bbbk \otimes D & = & D & = & D \otimes \Bbbk \end{array} \tag{24}$$

commutes, where ε is the counit on C. Observe that $\overline{f} \colon \overline{D} \to \overline{C}$ then makes $\overline{D}$ into a comodule over $\overline{C}$. The following proposition relates elements in $\mathrm{Coder}(\overline{D}, \overline{C})$ to coderivations $d \colon D \to C$ that satisfy $d(1) = 0$; the latter is called a *coderivation of coaugmented coalgebras.*

Proposition 20. *Let $f \colon D \to C$ be a homomorphism of coaugmented coalgebras. There is a one-to-one correspondence between coderivations $d \colon D \to C$ satisfying $d(1) = 0$ and coderivations $\overline{d} \colon \overline{D} \to \overline{C}$ given by $d = \overline{d} \oplus 0 \colon \overline{D} \oplus \Bbbk \to \overline{C} \oplus \Bbbk$.*

Proof. Given a linear map $\overline{d} \in \mathrm{Hom}(\overline{D}, \overline{C})$, one easily checks that $\overline{d}$ is a coderivation if and only if $\overline{d} \oplus 0 \colon \overline{D} \oplus \Bbbk \to \overline{C} \oplus \Bbbk$ is. It then suffices to show that each coderivation $d \colon D \to C$ with $d(1) = 0$ is of this form. Let ε be the counit of C and $\mu_\Bbbk \colon \Bbbk \otimes \Bbbk \to \Bbbk$ the multiplication on $\Bbbk$. From (13), (23) and the compatibility with the counit (24) it then follows that

$$\begin{aligned} \varepsilon \circ d &= \mu_\Bbbk(\varepsilon \otimes \varepsilon)\Delta_C \circ d \\ &= \mu_\Bbbk(\varepsilon \otimes (\varepsilon \circ d))\Delta_l + \mu_\Bbbk((\varepsilon \circ d) \otimes \varepsilon)\Delta_r \\ &= 2(\varepsilon \circ d), \end{aligned}$$

which shows that $d(D) \subset \overline{C}$. Hence, d decomposes as $\overline{d} \oplus 0 \colon \overline{D} \oplus \Bbbk \to \overline{C} \oplus \Bbbk$.

For a coassociative coalgebra C, we call an element $d \in \mathrm{Coder}(C)$ of degree one with $d^2 = 0$ a *codifferential* on C. We then call the pair (C, d) a *differential graded coassociative coalgebra* (*DGC* for short). If C is coaugmented and $d(1) = 0$, we call (C, d) a *coaugmented DGC.* A homomorphism of DGCs is then a coalgebra homomorphism that is also a homomorphism of DG vector spaces; homomorphisms of coaugmented DGCs are defined accordingly. From Proposition 20 and Section 2.4.1, we then obtain an equivalence between the categories of DGCs and coaugmented DGCs.

Proposition 21. *Let C be a coassociative coalgebra. Then $\mathrm{Coder}(C) \subset \mathfrak{gl}(C)$ is closed under the graded commutator. Also, if $f \colon D \to C$ is a homomorphism of coassociative coalgebras, $d \in \mathrm{Coder}(C)$ and $d' \in \mathrm{Coder}(D)$, then $f \circ d', d \circ f \in \mathrm{Coder}(D, C)$.*

Proof. Both parts of the proposition are straightforward computations which are left to the reader.

Theorem 22. *Let D be a cocommutative coaugmented coalgebra and $f \colon D \to \mathcal{S}(V)$ a homomorphism of coaugmented coalgebras. The linear map*

$$\mathrm{Coder}(D, \mathcal{S}(V)) \to \mathrm{Hom}(D, V), \quad d \mapsto \mathrm{pr}_V \circ d$$

is then an isomorphism. Its inverse is given by

$$\mathrm{Hom}(D, V) \to \mathrm{Coder}(D, \mathcal{S}(V)), \quad \lambda \mapsto \mu_S(\lambda \otimes f)\Delta_D,$$

where $\mu_S \colon \mathcal{S}(V) \otimes \mathcal{S}(V) \to \mathcal{S}(V)$ denotes the multiplication on $\mathcal{S}(V)$ and Δ_D the coproduct on D.

It is immediate that $d(1) = 0$ if and only if $\lambda = \mathrm{pr}_V \circ d$ vanishes on $\Bbbk$. Together with Proposition 20, this shows $\mathrm{Coder}(\overline{D}, \overline{\mathcal{S}}(V)) \cong \mathrm{Hom}(\overline{D}, V)$.

$$
\begin{aligned}
&\Delta_S(v_1)\dots\Delta_S(v_{n+1})\\
&=\Delta_S(v_1\vee\dots\vee v_n)\Delta_S(v_{n+1})\\
&=\sum_{i=0}^{n}\sum_{\tau\in\mathrm{Sh}^{-1}_{i,n-i}}\big(\varepsilon(\tau)(v_{\tau(1)}\vee\dots\vee v_{\tau(i)})\otimes(v_{\tau(i+1)}\vee\dots\vee v_{\tau(n)})\big)\,(v_{n+1}\otimes 1+1\otimes v_{n+1})\\
&=\sum_{i=0}^{n}\sum_{\sigma\in\mathrm{Sh}^{-1}_{i,n-i}}(-1)^{|v_{n+1}|\sum_{k=i+1}^{n}|v_{\sigma(k)}|}\varepsilon(\sigma)(v_{\sigma(1)}\vee\dots\vee v_{\sigma(i)}\vee v_{n+1})\otimes(v_{\sigma(i+1)}\vee\dots\vee v_{\sigma(n)})\\
&\quad+\sum_{i=0}^{n}\sum_{\sigma\in\mathrm{Sh}^{-1}_{i,n-i}}\varepsilon(\sigma)(v_{\sigma(1)}\vee\dots\vee v_{\sigma(i)})\otimes(v_{\sigma(i+1)}\vee\dots\vee v_{\sigma(n)}\vee v_{n+1})\\
&=\sum_{i=0}^{n+1}\sum_{\sigma\in\mathrm{Sh}^{-1}_{i,n+1-i}}\varepsilon(\sigma)(v_{\sigma(1)}\vee\dots\vee v_{\sigma(i)})\otimes(v_{\sigma(i+1)}\vee\dots\vee v_{\sigma(n+1)})\\
&=\Delta_S(v_1\vee\dots\vee v_{n+1}).
\end{aligned}
$$

The first part of Theorem 22 actually holds for a broader class of comodules over $\mathcal{S}(V)$ (see for example [8], Lemma 2.4); the inverse formula $d=\mu_S(\lambda\otimes\mathrm{id}_M)\Delta_r$ then continues to hold for comodules M in which $\tau_{M,S}\circ\Delta_r=\Delta_l$.

Remark 23. For $1\le i\le n-1$ and $\tau\in\mathrm{Sh}^{-1}_{i,n-i}$ either $\tau(i)=n$ or $\tau(n)=n$. In the first case, there is a unique $\sigma\in\mathrm{Sh}^{-1}_{i-1,n-i}$ such that $(\tau(1),\dots,\tau(n))=(\sigma(1),\dots,\sigma(i-1),n,\sigma(i),\dots,\sigma(n-1))$, while in the second case $(\tau(1),\dots,\tau(n))=(\sigma(1),\dots,\sigma(n-1),n)$ for a unique $\sigma\in\mathrm{Sh}^{-1}_{i,n-i-1}$. This yields a bijection $\mathrm{Sh}^{-1}_{i,n-i}\cong\mathrm{Sh}^{-1}_{i-1,n-i}\sqcup\mathrm{Sh}^{-1}_{i,n-i-1}$. By setting $\mathrm{Sh}^{-1}_{-1,n},\mathrm{Sh}^{-1}_{n,-1}=\emptyset$, this also holds for $i=0,n$.

Lemma 24. *The map $\Delta_S\colon\mathcal{S}(V)\to\mathcal{S}(V)\otimes\mathcal{S}(V)$ is a homomorphism of graded algebras.*

Proof. We show by induction over $n\in\mathbb{N}$ that

$$\Delta_S(v_1\vee\dots\vee v_n)=\Delta_S(v_1)\dots\Delta_S(v_n). \tag{25}$$

For $n=1$ there is nothing to do. Assume that (25) holds for $n\ge 1$. We compute

See this equation above.

In the fourth equality, we shifted the summation index of the first sum and used Remark 23.

Proof of Theorem 22. Let $d\colon D\to\mathcal{S}(V)$ be a coderivation, that is

$$\Delta_S\circ d=(d\otimes f+f\otimes d)\Delta_D.$$

Inductively, we then get

$$\Delta_S^n\circ d=\sum_{k=0}^{n}(f^{\otimes k}\otimes d\otimes f^{\otimes(n-k)})\Delta_D^n.$$

For $n\in\mathbb{N}$, let $\pi_n\colon T^n(V)\to\mathcal{S}^n(V)$ be the linear map defined by

$$\pi_n(v_1\otimes\dots\otimes v_n)=\frac{1}{n!}v_1\vee\dots\vee v_n.$$

From $\mathcal{S}(V)$ being a subcoalgebra of $T(V)$ and (17), it follows that $\pi_{n+1}\circ\mathrm{pr}_V^{\otimes(n+1)}\circ\Delta_S^n=\mathrm{pr}_{\mathcal{S}^{n+1}(V)}$. We then have

$$
\begin{aligned}
\mathrm{pr}_{\mathcal{S}^{n+1}(V)}\circ d&=\pi_{n+1}\circ\mathrm{pr}_V^{\otimes(n+1)}\\
&\quad\circ\sum_{k=0}^{n}(f^{\otimes k}\otimes d\otimes f^{\otimes(n-k)})\Delta_D^n\\
&=\pi_{n+1}\circ\sum_{k=0}^{n}((\mathrm{pr}_V\circ f)^{\otimes k}\otimes(\mathrm{pr}_V\circ d)\\
&\quad\otimes(\mathrm{pr}_V\circ f)^{\otimes(n-k)})\Delta_D^n.
\end{aligned}
$$

As this holds for all $n\in\mathbb{N}$ and as $\mathrm{pr}_{\mathbb{k}}\circ d=0$ by the same computation as in the proof of Proposition 20, d is completely determined by $\mathrm{pr}_V\circ d$.

What is left is to show that given $\lambda\in\mathrm{Hom}(D,V)$ homogeneous, $d:=\mu_S(\lambda\otimes f)\Delta_D$ is a coderivation with $\mathrm{pr}_V\circ d=\lambda$. While the latter holds by construction, we compute for $x\in D$ homogeneous

See this equation next page

where we used Lemma 24 in the first and cocommutativity of D in the fifth equality.

2.5 Dual spaces

The graded vector space $V^*:=\mathrm{Hom}(V,\mathbb{k})$ is called the *dual space of* V. By degree reasons, $(V^*)_k=\mathrm{Hom}_k(V,\mathbb{k})=\mathrm{Hom}(V_{-k},\mathbb{k})=(V_{-k})^*$. For $f\in\mathrm{Hom}_p(V,W)$, the linear map $f^*\in\mathrm{Hom}_p(W^*,V^*)$ is defined by $f^*(\varphi)=(-1)^{|\varphi||f|}\varphi\circ f$ for $\varphi\in W^*$ homogeneous. Note that

$$
\begin{aligned}
(\Delta_S \circ d)(x) &= \sum \Delta_S(\lambda(x_{(1)})) \vee \Delta_S(f(x_{(2)})) \\
&= \sum \big(\lambda(x_{(1)}) \otimes 1 + 1 \otimes \lambda(x_{(1)})\big) \vee (f \otimes f)(\Delta_D(x_{(2)})) \\
&= \sum \big(\lambda(x_{(1)}) \otimes 1 + 1 \otimes \lambda(x_{(1)})\big)(f(x_{(2)}) \otimes f(x_{(3)})) \\
&= \sum \big(\lambda(x_{(1)}) \vee f(x_{(2)})\big) \otimes f(x_{(3)}) + (-1)^{(|\lambda|+|x_{(1)}|)|x_{(2)}|} f(x_{(2)}) \otimes \big(\lambda(x_{(1)}) \vee f(x_{(3)})\big) \\
&= \sum \big(\mu_S(\lambda \otimes f) \otimes f) + f \otimes \mu_S(\lambda \otimes f)\big)(x_{(1)} \otimes x_{(2)} \otimes x_{(3)}) \\
&= \big(\mu_S(\lambda \otimes f)\Delta_D \otimes f + f \otimes \mu_S(\lambda \otimes f)\Delta_D\big)(\Delta_D(x)),
\end{aligned}
$$

$\mathrm{id}_V^* = \mathrm{id}_{V^*}$ and if g is a homogeneous linear map with domain W, we have $(g \circ f)^* = (-1)^{|f||g|} f^* \circ g^*$.

We say that V is *of finite type* if V_k is finite-dimensional for all $k \in \mathbb{Z}$. Note that if V is of finite type, the canonical inclusion $V \hookrightarrow V^{**}$ is an isomorphism.

If $V_k = 0$ for $k > 0$, then V is called *$\mathbb{Z}_{\leq 0}$-graded*. Notions as *$\mathbb{Z}_{<0}$-graded* or *$\mathbb{Z}_{\geq 0}$-graded* are defined accordingly. In the following, we denote $V^{\otimes n}$ as $T^n(V)$ for better readability.

Proposition 25. *If V is of finite type and if for all $k \in \mathbb{Z}$ the decomposition*

$$T^n(V)_k = \bigoplus_{i_1+\ldots+i_n=k} (V_{i_1} \otimes \ldots \otimes V_{i_n})$$

has only finitely many non-trivial summands, then the canonical inclusion $T^n(V^*) \hookrightarrow T^n(V)^*$ is an isomorphism.

Proof. It is well-known that for finite-dimensional (ungraded) vector spaces $V_1, \ldots, V_n$, the canonical inclusion $V_1^* \otimes \ldots \otimes V_n^* \hookrightarrow (V_1 \otimes \ldots \otimes V_n)^*$ is an isomorphism. We then have

$$
\begin{aligned}
(T^n(V)_{-k})^* &= \bigoplus_{i_1+\ldots+i_n=k} (V_{-i_1} \otimes \ldots \otimes V_{-i_n})^* \\
&\cong \bigoplus_{i_1+\ldots+i_n=k} (V^*)_{i_1} \otimes \ldots \otimes (V^*)_{i_n} \\
&= T^n(V^*)_k.
\end{aligned}
$$

Remark 26. If V is of finite type and $\mathbb{Z}_{\leq 0}$-graded, V^* is also of finite type and $\mathbb{Z}_{\geq 0}$-graded and they both satisfy the hypothesis of Proposition 25. It is then easy to see that $\Delta\colon V \to V \otimes V$ makes V into a graded coassociative/cocommutative coalgebra if and only if $\Delta^*\colon V^* \otimes V^* \cong (V \otimes V)^* \to V^*$ makes V^* into an associative/commutative algebra. A linear map $d\colon V \to V$ is then a coderivation of (V, Δ) if and only if $-d^*$ is a derivation of (V^*, Δ^*). The map $\mathfrak{gl}(V) \to \mathfrak{gl}(V^*)$, $f \mapsto -f^*$ preserves the graded commutator and therefore restricts to an isomorphism of graded Lie algebras $\mathrm{Coder}(V) \cong \mathrm{Der}(V^*)$.

Corollary 27. *If V is of finite type and $\mathbb{Z}_{<0}$-graded, the canonical inclusion $T(V^*) \hookrightarrow T(V)^*$ is an isomorphism.*

Proof. Note that for all $k \in \mathbb{Z}$ and $n > k$, we have $T^n(V^*)_{-k} = 0$. Then

$$
\begin{aligned}
(T(V)_{-k})^* &\cong \bigoplus_{n \geq 0} (T^n(V)_{-k})^* = \bigoplus_{n \geq 0} (T^n(V)^*)_k \\
&\cong \bigoplus_{n \geq 0} T^n(V^*)_k = T(V^*)_k.
\end{aligned}
$$

Lemma 28. *Let $\xi \in T^n(V^*) \subset T^n(V)^*$ and $\sigma \in \mathfrak{S}_n$. Then*

$$\hat{\varepsilon}(\sigma)\xi = \xi \circ \hat{\varepsilon}(\sigma^{-1}). \tag{26}$$

Proof. It suffices to show this for $\sigma = s_1, \ldots, s_{n-1}$, in which case it is an easy computation.

Proposition 29. *If V is of finite type and $\mathbb{Z}_{<0}$-graded, $\mathcal{S}(V)^* \cong \mathcal{S}(V^*)$. Under this identification, Δ_S^* is the usual multiplication on $\mathcal{S}(V^*)$.*

Proof. Fix $n \geq 0$. By Lemma 28, the isomorphism $T^n(V^*) \cong T^n(V)^*$ maps the subspace of symmetric elements in $T^n(V^*)$ onto the space of symmetric linear maps $V^{\otimes n} \to \Bbbk$. While the latter is isomorphic to $\mathrm{Hom}(\mathcal{S}^n(V), \Bbbk) \cong \mathcal{S}^n(V)^*$ by Remark 2, the former is isomorphic to $\mathcal{S}^n(V^*)$ via the linear map

$$
\begin{aligned}
&\mathcal{S}^n(V^*) \to T^n(V^*), \\
&v_1 \vee \ldots \vee v_n \mapsto \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}.
\end{aligned}
$$

This yields an isomorphism $\mathcal{S}^n(V^*) \cong \mathcal{S}^n(V)^*$. With a similar reasoning as in the proof of Corollary 27, one obtains $\mathcal{S}(V^*) \cong \mathcal{S}(V)^*$. It is then a straightforward computation to show that Δ_S^* is indeed the usual multiplication on $\mathcal{S}(V^*)$.

3 L_∞-algebras

We start this section with a theorem from [5] that characterises certain L_∞-algebras using Lie algebra cohomology; later, we seek to generalise it in the context of L_∞-algebra cohomology. After that, we discuss different characterisations of L_∞-structures using the key results of Section 2. Different points of view naturally lead to different notions of homomorphisms between L_∞-algebras; we will finish the section with a comparison of those. For this, we will mostly

$$\begin{aligned} l_1(l_1(x_1)) &= 0, \\ l_1(l_2(x_1,x_2)) &= l_2(l_1(x_1),x_2) + (-1)^{|x_1|} l_2(x_1, l_1(x_2)), \\ 0 &= \sum_{\sigma \in \mathrm{Sh}_{2,1}^{-1}} \chi(\sigma) l_2(l_2(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) + l_1(l_3(x_1,x_2,x_3)) + l_3(l_1(x_1),x_2,x_3) \\ &\qquad + (-1)^{|x_1|} l_3(x_1, l_1(x_2), x_3) + (-1)^{|x_1|+|x_2|} l_3(x_1,x_2,l_1(x_3)) \end{aligned}$$

follow [3], although the original references for Section 3.2 are [1] and [8].

Definition 30. An L_∞-*algebra* is a graded vector space L together with antisymmetric linear maps $l_k\colon L^{\otimes k} \to L$ called *(higher) brackets* of degree $|l_k| = 2-k$ for $1 \le k < \infty$ such that the *generalized Jacobi identity*

$$\sum_{i+j=n+1} \sum_{\sigma \in \mathrm{Sh}_{i,n-i}^{-1}} (-1)^{i(j-1)} \chi(\sigma) \cdot l_j(l_i(x_1,\dots,x_i), x_{i+1},\dots,x_n) = 0 \tag{27}$$

holds for all $n \ge 1$ and $x_1,\dots,x_n \in L$ homogeneous. We then call the set $\{l_k \mid 1 \le k < \infty\}$ an L_∞-*structure* on L.

Writing out (27) for $n = 1, 2, 3$ yields

See this equation above

for all $x_1, x_2, x_3 \in L$ homogeneous. While the first two equations may be summarized by saying that l_1 is a differential on the (non-associative) graded algebra (L, l_2), a comparison with (11) shows that the third one describes the defect of the Jacobi identity in (L, l_2). In particular, an L_∞-algebra with $l_k = 0$ for $k \ge 3$ is nothing else than a DGLA.

If L is concentrated in degree zero, $l_k = 0$ for $k \ne 2$ by degree reasons and (L, l_2) is an (ungraded) Lie algebra.

Definition 31. Let L and L' be L_∞-algebras with L_∞-structures $\{l_k\}_{k\in\mathbb{N}}$ and $\{l'_k\}_{k\in\mathbb{N}}$, respectively. A *strict L_∞-algebra homomorphism* is a degree preserving linear map $f\colon L \to L'$ satisfying

$$f \circ l_k = l'_l \circ f^{\otimes k} \tag{28}$$

for all $1 \le k < \infty$.

These homomorphisms are strict in the sense that they strictly preserve all brackets. A different characterisation of L_∞-algebras will later lead to a more general notion of L_∞-algebra homomorphisms.

3.1 Characterisation via Lie algebra cohomology

For an (ungraded) Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$, a *representation* of $\mathfrak{g}$ on an (ungraded) vector space V is a homomorphism of Lie algebras $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$. Given such a ρ, the *Lie algebra cohomology with values in V* is the cohomology of the *Chevalley–Eilenberg (cochain) complex* $\big(\bigoplus_{n\ge 0} \mathrm{Hom}(\bigwedge^n \mathfrak{g}, V), \delta\big)$, where for an antisymmetric linear map $\omega\colon \mathfrak{g}^{\otimes n} \to V$, we define $\delta\omega$ by

$$\begin{aligned} \delta\omega(x_1,\dots,x_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} \rho(x_i)(\omega(x_1,\dots,\hat{x}_i,\dots,x_{n+1})) \\ &\quad + \sum_{1\le j<k\le n+1} (-1)^{j+k} \\ &\quad \times \omega([x_j,x_k], x_1,\dots,\hat{x}_j,\dots,\hat{x}_k,\dots,x_{n+1}), \end{aligned}$$

for $x_1,\dots,x_{n+1} \in \mathfrak{g}$. In the sums above, elements with $\hat{}$ are to be omitted.

Theorem 32 ([5], Theorem 55)**.** *There is for each $n \ge 1$ a one-to-one correspondence between L_∞-algebras L such that $L_k = 0$ for $k \ne -n, 0$ and $l_1 = 0$ and quadruples $(\mathfrak{g}, V, \rho, l_{n+2})$ consisting of a Lie algebra $\mathfrak{g}$, a representation ρ of $\mathfrak{g}$ on a vector space V and an $(n+2)$-cocycle l_{n+2}.*

Sketch of proof. For an L_∞-algebra $L = L_0 \oplus L_{-n}$ with $l_1 = 0$, all brackets exept for l_2 and l_{n+2} have to vanish by degree reasons. Also, l_2 has to vanish on $\bigwedge^2 L_{-n}$ and l_{n+2} can only be non-trivial on $\bigwedge^{n+2} L_0$ with image in L_{-n}. Using (8), we can decompose l_2 into linear maps $[\cdot,\cdot]\colon \bigwedge^2 L_0 \to L_0$ and $\rho\colon L_0 \otimes L_{-n} \to L_{-n}$. It is then a matter of computation to show that l_2 and l_{n+2} satisfying (27) amounts to $(L_0, [\cdot,\cdot])$ being a Lie algebra, ρ being a representation of L_0 on L_{-n} and l_{n+2} being a cocycle.

3.2 Symmetric brackets and codifferentials

Recall that by Corollary 4, an antisymmetric map $l_k\colon L^{\otimes k} \to L$ of degree $2-k$ is equivalent to a symmetric degree one map $\lambda_k\colon \mathcal{S}^k(L[1]) \to L[1]$ such that

$$l_k = \uparrow \circ \lambda_k \circ \downarrow^{\otimes k}.$$

If we now rewrite (27) in terms of the maps λ_k, we obtain a characterisation of L_∞-structures on L in terms of symmetric brackets. Note that for fixed n, we can write (27) as

$$\sum_{i+j=n+1} \sum_{\sigma \in \mathrm{Sh}_{i,n-i}^{-1}} (-1)^{(j-1)i} l_j(l_i \otimes \mathrm{id}_L^{\otimes(j-1)}) \hat{\chi}(\sigma) = 0$$

As $\downarrow$ and $\uparrow^{\otimes n}$ are isomorphisms, this is equivalent to

See this equation next page

where we used (2) and (4). We have proved the following.

$$
\begin{aligned}
0 &= (-1)^{\frac{n(n-1)}{2}} \sum_{i+j=n+1} \sum_{\sigma\in \mathrm{Sh}_{i,n-i}^{-1}} (-1)^{(j-1)i} \downarrow \circ\, l_j(l_i \otimes \mathrm{id}_L^{\otimes(j-1)})\hat{\chi}(\sigma) \circ \uparrow^{\otimes n} \\
&= (-1)^{\frac{n(n-1)}{2}} \sum_{i+j=n+1} \sum_{\sigma\in \mathrm{Sh}_{i,n-i}^{-1}} (-1)^{(j-1)i} \lambda_j \circ \downarrow^{\otimes j}(l_i \otimes \mathrm{id}_L^{\otimes(j-1)})\uparrow^{\otimes n} \circ\, \hat{\varepsilon}(\sigma) \\
&= (-1)^{\frac{n(n-1)}{2}} \sum_{i+j=n+1} \sum_{\sigma\in \mathrm{Sh}_{i,n-i}^{-1}} \lambda_j((\downarrow \circ\, l_i) \otimes \downarrow^{\otimes(j-1)})\uparrow^{\otimes n} \circ\, \hat{\varepsilon}(\sigma) \\
&= \sum_{i+j=n+1} \sum_{\sigma\in \mathrm{Sh}_{i,n-i}^{-1}} \lambda_j(\lambda_i \otimes \mathrm{id}_{L[1]}^{\otimes(j-1)})\hat{\varepsilon}(\sigma),
\end{aligned}
$$

Proposition 33. *An L_∞-structure $\{l_k \mid 1 \le k < \infty\}$ on the graded vector space L is equivalent to a system of linear maps $\lambda_k \colon \mathcal{S}^k(L[1]) \to L[1]$ for $1 \le k < \infty$, all of degree one, such that*

$$
\sum_{i+j=n+1} \sum_{\sigma\in \mathrm{Sh}_{i,n-i}^{-1}} \varepsilon(\sigma) \cdot \lambda_j(\lambda_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0 \tag{29}
$$

holds for all for all $n \ge 1$ and $x_1, \dots, x_n \in L[1]$ homogeneous.

Corollary 34. *An L_∞-structure on the graded vector space L is equivalent to a linear degree one map $\lambda \colon \overline{\mathcal{S}}(L[1]) \to L[1]$ such that*

$$
\lambda \circ \mu_S(\lambda \otimes \mathrm{id}_S)\Delta_S = 0. \tag{30}
$$

Proof. Combine the brackets in Proposition 33 to a single element $\lambda = \sum_k \lambda_k \in \mathrm{Hom}(\overline{\mathcal{S}}(L[1]), L[1])$ of degree one and compare with (18).

By abuse of notation, we then also refer to the pair $(L[1], \lambda)$ as an L_∞-algebra. Strict homomorphisms of L_∞-algebras can then be described as degree preserving linear maps that preserve the symmetric brackets.

Proposition 35. *Let $(L[1], \lambda)$ and $(L'[1], \lambda')$ be L_∞-algebras. There is a one-to-one correspondence between strict L_∞-algebra homomorphisms $f \colon L \to L'$ and linear degree preserving maps $g = \downarrow \circ f \circ \uparrow \colon L[1] \to L'[1]$ satisfying*

$$
g \circ \lambda_k = \lambda'_k \circ g^{\otimes k} \tag{31}
$$

for all $k \ge 1$.

The equation (31) can be written more conveniently as

$$
g \circ \lambda = \lambda' \circ \mathcal{S}(g). \tag{32}
$$

We then also refer to g as a strict L_∞-algebra homomorphism.

Theorem 36. *An L_∞-structure on the graded vector space L is equivalent to a codifferential $d \in \mathrm{Coder}(\mathcal{S}(L[1]))$ with $d(1) = 0$.*

In this case, we also refer to the pair $(\mathcal{S}(L[1]), d)$ as an L_∞-algebra.

Proof. Let $\lambda \colon \overline{\mathcal{S}}(L[1]) \to L[1]$ be of degree one and $d = \mu_S(\lambda \otimes \mathrm{id}_S)\Delta_S$ be the unique coderivation extending λ in the sense of Theorem 22. As $d^2 = \frac{1}{2}[d, d]$ is a coderivation of $\mathcal{S}(L[1])$ by Proposition 21, we have by Theorem 22 that $d^2 = 0$ if and only if

$$
0 = \mathrm{pr}_{L[1]} \circ d^2 = \lambda \circ \mu_S(\lambda \otimes \mathrm{id}_S)\Delta_S.
$$

Corollary 37. *If L is of finite type and $\mathbb{Z}_{\le 0}$-graded, an L_∞-structure on L is equivalent to a differential on the graded algebra $\mathcal{S}(L[1]^*)$. Explicitly, consider $d_{CE} = -d^*$ for d as in Theorem 36.*

Proof. See Remark 26 and note that as L is $\mathbb{Z}_{\le 0}$-graded, each $d \in \mathrm{Coder}(\mathcal{S}(L[1]))$ of degree one vanishes on $\Bbbk$ by degree reasons.

3.3 Weak homomorphisms

Let $(\mathcal{S}(L[1]), d)$ and $(\mathcal{S}(L'[1]), d')$ be L_∞-algebras, $\lambda = \mathrm{pr}_{L[1]} \circ d$ and $\lambda' = \mathrm{pr}_{L'[1]} \circ d'$. The characterisation of L_∞-structures as codifferentials on the symmetric coalgebra leads to another notion of homomorphisms of L_∞-algebras, namely as homomorphisms of (coaugmented) DGCs.

Definition 38. A *(weak) homomorphism of L_∞-algebras between L and L'* is a homomorphism of coaugmented DGCs $f \colon \mathcal{S}(L[1]) \to \mathcal{S}(L'[1])$.

Remark 39. By Proposition 21 and Theorem 22, a homomorphism of coaugmented coalgebras $f \colon \mathcal{S}(L[1]) \to \mathcal{S}(L'[1])$ is a homomorphism of L_∞-algebras if and only if

$$
(\mathrm{pr}_{L'[1]} \circ f) \circ d = \lambda' \circ f.
$$

Note that by Proposition 20, it makes sense to also refer to DGC homomorphisms $\overline{\mathcal{S}}(L[1]) \to \overline{\mathcal{S}}(L'[1])$ as homomorphisms of L_∞-algebras.

From the dualised standpoint, we immediately get the following.

Proposition 40. *Assume that L and L' are $\mathbb{Z}_{\le 0}$-graded and of finite type. Then $f \colon \mathcal{S}(L[1]) \to \mathcal{S}(L'[1])$ is a homomorphism of L_∞-algebras if and only if $f^* \colon \mathcal{S}(L'[1]^*) \to \mathcal{S}(L[1]^*)$ is a homomorphism of unital DG algebras.*

With now two different notions of L_∞-algebra homomorphisms at hand, it is reasonable to ask if there is a connection between them. As commented in ([8], Remark 5.3), strict homomorphisms are essentially the weak homomorphisms that preserve the exterior degree.

Lemma 41. *Let $g\colon L[1] \to L'[1]$ be a linear degree preserving map. Then g is a strict L_∞-algebra homomorphism if and only if $\mathcal{S}(g)$ is a weak one.*

Proof. Observe that $\mathrm{pr}_{L'[1]} \circ \mathcal{S}(g) \circ d = g \circ \mathrm{pr}_{L[1]} \circ d = g \circ \lambda$.

Lemma 42. *A homomorphism of coalgebras $f\colon \mathcal{S}(L[1]) \to \mathcal{S}(L'[1])$ preserves the exterior degree if and only if $f = \mathcal{S}(g)$ for a linear degree preserving map $g\colon L[1] \to L'[1]$.*

Proof. Assume that $f\colon \mathcal{S}(L[1]) \to \mathcal{S}(L'[1])$ is a homomorphism of coalgebras such that $f(\mathcal{S}^n(L[1])) \subset \mathcal{S}^n(L'[1])$ for all n and let $g := \mathrm{pr}_{L'[1]} \circ f|_{L[1]}$. Then $\mathrm{pr}_{L'[1]} \circ f = g \circ \mathrm{pr}_{L[1]} = \mathrm{pr}_{L'[1]} \circ \mathcal{S}(g)$. Hence, $f = \mathcal{S}(g)$ by Proposition 18.

Proposition 43. *Let $f\colon \mathcal{S}(L[1]) \to \mathcal{S}(L'[1])$ be a (weak) L_∞-algebra homomorphism. Then f preserves the exterior degree if and only if $f = \mathcal{S}(g)$ for a strict L_∞-algebra homomorphism g.*

Proof. Combine Lemma 41 and Lemma 42.

From this it follows for example that all (weak) L_∞-algebra homomorphisms between Lie algebras are induced by Lie algebra homomorphisms.

4 Representations (up to homotopy)

While representations (up to homotopy) of L_∞-algebras are often defined in terms of antisymmetric maps, we start with a definition that keeps the symmetric point of view of the last section. While it is a straightforward computation to show equivalence between these definitions, it is convenient to save this for Section 5.1. We then show that representations (up to homotopy) are nothing else than weak L_∞-algebra homomorphisms into $\mathfrak{gl}(V)$ for a DG vector space V, a characterisation due to Lada and Markl [8]. In [3], representations (up to homotopy) were described (under some finiteness assumptions) as differentials on $\mathcal{S}(L[1]^*) \otimes V$. We discuss this point of view in the second half of this section, which also leads us to L_∞-algebra cohomology.

From now on, L denotes an L_∞-algebra with L_∞-structure $\{l_k \mid 1 \le k < \infty\}$ and λ and d are as in Corollary 34 and Theorem 36, respectively.

Definition 44. A *representation (up to homotopy) of L on V* is a linear map $\rho\colon \mathcal{S}(L[1]) \otimes V \to V$ of degree one that satisfies

$$\rho(d \otimes \mathrm{id}_V) + \rho(\mathrm{id}_S \otimes \rho)(\Delta_S \otimes \mathrm{id}_V) = 0. \qquad (33)$$

4.1 Representations as (weak) homomorphisms

We prove the following version of ([8], Theorem 5.2).

Theorem 45. *There is a one-to-one correspondence between representations of L on V and pairs $(\partial, \tilde{f})$, where ∂ is a differential on V and $\tilde{f}\colon \mathcal{S}(L[1]) \to \mathcal{S}(\mathfrak{gl}(V)[1])$ a homomorphism of L_∞-algebras. Here, $\mathfrak{gl}(V)$ carries the DGLA structure induced by ∂, see Example 12.*

One should therefore really think of L being represented on a DG vector space. The following lemma characterises L_∞-algebra homomorphisms into DGLAs and is a symmetric version of ([8], Definition 5.2).

Lemma 46. *Let (L', l'_2, l'_1) be a DGLA and $\lambda' = \lambda'_1 + \lambda'_2$ be the corresponding linear degree one map $\overline{\mathcal{S}}(L'[1]) \to L'[1]$. For $f\colon \overline{\mathcal{S}}(L[1]) \to L'[1]$ a linear degree preserving map, the induced homomorphism of coalgebras $\tilde{f}\colon \overline{\mathcal{S}}(L[1]) \to \overline{\mathcal{S}}(L'[1])$ is a homomorphism of L_∞-algebras if and only if*

$$f \circ \overline{d} = \lambda'_1 \circ f + \frac{1}{2}\lambda'_2(f \otimes f)\overline{\Delta}_S. \qquad (34)$$

This is the case if and only if the linear degree one map $\rho\colon \overline{\mathcal{S}}(L[1]) \to L'$ defined by $f(x) = (-1)^{|x|+1}{\downarrow}\rho(x)$ satisfies

$$\rho \circ \overline{d} + l'_1 \circ \rho + \frac{1}{2} l'_2(\rho \otimes \rho)\overline{\Delta}_S = 0. \qquad (35)$$

Proof. The first part follows immediately from Remark 39 and the explicit construction of $\tilde{f}$ (see Proposition 18). It is straightforward to check that for $x \in \overline{\mathcal{S}}(L[1])$ homogeneous,

$$\begin{aligned} f(d(x)) &= (-1)^{|x|}{\downarrow}\rho(d(x)), \\ \lambda'_1(f(x)) &= (-1)^{|x|+1}{\downarrow} l'_1(\rho(x)), \\ \big(\lambda'_2(f \otimes f)\overline{\Delta}_S\big)(x) &= (-1)^{|x|+1}{\downarrow}\big(l'_2(\rho \otimes \rho)\overline{\Delta}_S\big)(x), \end{aligned}$$

from which the second part then follows.

Proof of Theorem 45: As $\mathrm{Hom}(\mathcal{S}(L[1]) \otimes V, V) \cong \mathrm{Hom}(\mathcal{S}(L[1]), \mathfrak{gl}(V))$, a linear degree one map $\rho\colon \mathcal{S}(L[1]) \otimes V \to V$ can be decomposed into linear degree one maps $\tilde{\rho}\colon \overline{\mathcal{S}}(L[1]) \to \mathfrak{gl}(V)$ and $\rho_0\colon \Bbbk \to \mathfrak{gl}(V)$; the latter being equivalent to the choice of a degree one element $\partial = \rho_0(1_\Bbbk) \in \mathfrak{gl}(V)$. If we show that under this identification ρ satisfying (33) is equivalent to $\partial^2 = 0$ and $\tilde{\rho}$ satisfying (35), the assertion follows by Lemma 46. For $x \in \overline{\mathcal{S}}(L[1])$ homogeneous,

$$\begin{aligned} &\frac{1}{2}\big([\cdot,\cdot](\tilde{\rho} \otimes \tilde{\rho})\overline{\Delta}_S\big)(x) \\ &\quad= \frac{1}{2}\sum (-1)^{|x_{(1)}|}[\tilde{\rho}(x_{(1)}), \tilde{\rho}(x_{(2)})] \\ &\quad= \frac{1}{2}\sum (-1)^{|x_{(1)}|}\tilde{\rho}(x_{(1)}) \circ \tilde{\rho}(x_{(2)}) \\ &\qquad + (-1)^{|x_{(1)}||x_{(2)}| + |x_{(2)}|}\tilde{\rho}(x_{(2)}) \circ \tilde{\rho}(x_{(1)}) \\ &\quad= \frac{1}{2}\sum \rho(\mathrm{id}_S \otimes \rho)(x_{(1)} \otimes x_{(2)} \\ &\qquad + (-1)^{|x_{(1)}||x_{(2)}|} x_{(2)} \otimes x_{(1)}, \cdot\,) \\ &\quad= \rho(\mathrm{id}_S \otimes \rho)(\overline{\Delta}_S(x), \cdot\,) \end{aligned}$$

by cocommutativity of $\overline{\mathcal{S}}(L[1])$ and

$$\begin{aligned}[\partial, \tilde{\rho}(x)] &= \partial \circ \tilde{\rho}(x) - (-1)^{|\tilde{\rho}(x)|}\tilde{\rho}(x) \circ \partial \\ &= \rho_0(1) \circ \tilde{\rho}(x) + (-1)^{|x|}\tilde{\rho}(x) \circ \rho_0(1) \\ &= \rho(\mathrm{id}_S \otimes \rho)(1 \otimes x + x \otimes 1, \cdot).\end{aligned}$$

As $(\tilde{\rho} \circ d)(x) = \rho(d \otimes \mathrm{id}_V)(x, \cdot)$, $\tilde{\rho}$ satisfying (35) is equivalent to (33) holding on $\overline{\mathcal{S}}(L[1]) \otimes V$. We also have

$$\begin{aligned}&\big(\rho(d \otimes \mathrm{id}_V) + \rho(\mathrm{id}_S \otimes \rho)(\Delta_S \otimes \mathrm{id}_V)\big)(1, \cdot) \\ &\quad = \rho(1, \rho(1, \cdot)) = \partial^2,\end{aligned}$$

which completes the proof as $\mathcal{S}(L[1]) = \Bbbk \oplus \overline{\mathcal{S}}(L[1])$.

Example 47 (The trivial representation on a DG vector space)**.** Let (V, ∂) be a DG vector space. There is a trivial strict homomorphism of L_∞-algebras $0\colon L \to \mathfrak{gl}(V)$. The induced representation $\mathcal{S}(L[1]) \otimes V \to V$ is on $\Bbbk \otimes V \cong V$ given by ∂ and zero elsewhere and is called the *trivial representation* of L on V. In particular, there is a trivial representation of L on $\Bbbk$.

Remark 48. Let ρ be a representation of L on V and $(\partial, \tilde{f})$ as in Theorem 45.

(1) Then $-\partial^*$ is a differential on V^* and the map $\mathfrak{gl}(V) \to \mathfrak{gl}(V^*)$, $g \mapsto -g^*$ is a homomorphism of DGLAs. By composing the corresponding weak homomorphism with $\tilde{f}$, we obtain an L_∞-algebra homomorphism $\mathcal{S}(L[1]) \to \mathcal{S}(\mathfrak{gl}(V^*)[1])$. The induced representation is given by

$$\rho^\vee\colon \mathcal{S}(L[1]) \otimes V^* \to V^*, \quad x \otimes \xi \mapsto -\rho(x, \cdot)^*\xi$$

and is called the representation *dual to* ρ.

(2) Fix $n \in \mathbb{Z}$. Then $(-1)^n {\downarrow}^n \circ \partial \circ {\uparrow}^n$ is a differential on $V[n]$ and

$$\mathfrak{gl}(V) \to \mathfrak{gl}(V[n]), \quad g \mapsto (-1)^{n|g|}{\downarrow}^n \circ g \circ {\uparrow}^n$$

is a DGLA homomorphism. The induced representation of L on $V[n]$ is given by

$$\begin{aligned}&\mathcal{S}(L[1]) \otimes V[n] \to V[n], \\ &x \otimes {\downarrow}^n v \mapsto (-1)^{n+n|x|}{\downarrow}^n \rho(x, v).\end{aligned}$$

4.2 Representations as coderivations

Observe that the map $\Delta_V := \Delta_S \otimes \mathrm{id}_V\colon \mathcal{S}(L[1]) \otimes V \to \mathcal{S}(L[1]) \otimes (\mathcal{S}(L[1]) \otimes V)$ satisfies

$$(\Delta_S \otimes \mathrm{id}_{S\otimes V})\Delta_V = (\mathrm{id}_S \otimes \Delta_V)\Delta_V, \tag{36}$$

which makes $\mathcal{S}(L[1]) \otimes V$ into a left $\mathcal{S}(L[1])$-comodule.

Definition 49. Let $d' \in \mathrm{Coder}(\mathcal{S}(L[1]))$ be of degree p. A *coderivation of* $\mathcal{S}(L[1]) \otimes V$ *extending* d' is a linear map $D\colon \mathcal{S}(L[1]) \otimes V \to \mathcal{S}(L[1]) \otimes V$ of degree p such that

$$\Delta_V \circ D = (d' \otimes \mathrm{id}_{S\otimes V} + \mathrm{id}_S \otimes D)\Delta_V. \tag{37}$$

Proposition 50 ([6], Proposition 1.5.3, p. 31)**.** *Let $d' \in \mathrm{Coder}(\mathcal{S}(L[1]))$ be of degree p. There is a one-to-one correspondence between coderivations D of $\mathcal{S}(L[1]) \otimes V$ extending d' and linear maps $\rho\colon \mathcal{S}(L[1]) \otimes V \to V$ of degree p given by*

$$\begin{aligned}D &= d' \otimes \mathrm{id}_V + (\mathrm{id}_S \otimes \rho)\Delta_V, \\ \rho &= \mathrm{pr}_V \circ D,\end{aligned}$$

where $\mathrm{pr}_V\colon \mathcal{S}(L[1]) \otimes V \to V$ is the projection of $\mathcal{S}(L[1]) \otimes V$ onto $\Bbbk \otimes V \cong V$.

Proof. Let D be a coderivation of $\mathcal{S}(L[1]) \otimes V$ extending d'. As $(\mathrm{id}_S \otimes \mathrm{pr}_V)(\Delta_S \otimes \mathrm{id}_V) = \mathrm{id}_S \otimes \mathrm{id}_V$, we obtain from (37) that

$$\begin{aligned}D &= (\mathrm{id}_S \otimes \mathrm{pr}_V)\Delta_V \circ D \\ &= (\mathrm{id}_S \otimes \mathrm{pr}_V)(d' \otimes \mathrm{id}_S \otimes \mathrm{id}_V + \mathrm{id}_S \otimes D)\Delta_V \\ &= (d' \otimes \mathrm{id}_V)(\mathrm{id}_S \otimes \mathrm{pr}_V)\Delta_V + (\mathrm{id}_S \otimes (\mathrm{pr}_V \circ D))\Delta_V \\ &= d' \otimes \mathrm{id}_V + (\mathrm{id}_S \otimes (\mathrm{pr}_V \circ D))\Delta_V.\end{aligned}$$

This shows that D is completely determined by $\mathrm{pr}_V \circ D$.

Let conversely $\rho \in \mathrm{Hom}(\mathcal{S}(L[1]) \otimes V, V)$ be of degree p. Using (36) and that d' is a coderivation, we compute

$$\begin{aligned}\Delta_V \circ (\mathrm{id}_S \otimes \rho)\Delta_V &= (\Delta_S \otimes \rho)\Delta_V \\ &= (\mathrm{id}_{S\otimes S} \otimes \rho)(\mathrm{id}_S \otimes \Delta_V)\Delta_V \\ &= (\mathrm{id}_S \otimes (\mathrm{id}_S \otimes \rho)\Delta_V)\Delta_V, \\ \Delta_V \circ (d' \otimes \mathrm{id}_V) &= ((d' \otimes \mathrm{id}_S + \mathrm{id}_S \otimes d')\Delta_S) \otimes \mathrm{id}_V \\ &= (d' \otimes \mathrm{id}_{S\otimes V} + \mathrm{id}_S \otimes d' \otimes \mathrm{id}_V)\Delta_V,\end{aligned}$$

which, combined, show that $D := d \otimes \mathrm{id}_V + (\mathrm{id}_S \otimes \rho)\Delta_V$ is a coderivation of $\mathcal{S}(L[1]) \otimes V$ extending d. It is easy to see that then $\mathrm{pr}_V \circ D = \rho$, which completes the proof.

Corollary 51. *There is a one-to-one correspondence between representations (up to homotopy) of L on V and coderivations $D\colon \mathcal{S}(L[1]) \otimes V \to \mathcal{S}(L[1]) \otimes V$ extending d such that $D^2 = 0$.*

Proof. It is a straightforward computation to check that $D^2 = \frac{1}{2}[D, D]$ is a coderivation of $\mathcal{S}(L[1]) \otimes V$ extending $\frac{1}{2}[d, d] = d^2 = 0$. By Proposition 50, $D^2 = 0$ if and only if $\rho = \mathrm{pr}_V \circ D$ satisfies

$$0 = \mathrm{pr}_V \circ D^2 = \rho(d \otimes \mathrm{id}_V) + \rho(\mathrm{id}_S \otimes \rho)(\Delta_S \otimes \mathrm{id}_V).$$

4.3 A first approach to L_∞-algebra cohomology

Assume now that the L_∞-algebra L is $\mathbb{Z}_{\leq 0}$-graded and of finite type and that V is either finite-dimensional or of finite type and trivial in the negative degrees. We then have $\mathcal{S}(L[1])^* \cong \mathcal{S}(L[1]^*)$, $V \cong V^{**}$ and $(\mathcal{S}(L[1]) \otimes V^*)^* \cong \mathcal{S}(L[1]^*) \otimes V$. Let $d_{CE} = -d^*$ denote the differential on $\mathcal{S}(L[1]^*)$. The map

$$\begin{aligned}&\mathcal{S}(L[1]^*) \otimes (\mathcal{S}(L[1]^*) \otimes V) \to \mathcal{S}(L[1]^*) \otimes V, \\ &\quad (\xi \otimes (\eta \otimes v)) \mapsto (\xi \vee \eta) \otimes v\end{aligned}$$

makes $\mathcal{S}(L[1]^*) \otimes V$ into a left $\mathcal{S}(L[1]^*)$-module. Similarly to Definition 49, we call a linear map $D_{CE}\colon \mathcal{S}(L[1]^*) \otimes$

$V \to \mathcal{S}(L[1]^*) \otimes V$ of degree one a *derivation of* $\mathcal{S}(L[1]^*) \otimes V$ *extending* d_{CE} if

$$D_{CE}(\xi \vee (\eta \otimes v)) = d_{CE}\xi \vee (\eta \otimes v) + (-1)^{|\xi|}\xi \vee D_{CE}(\eta \otimes v)$$

holds for all $\xi, \eta \in \mathcal{S}(L[1]^*)$, $v \in V$ homogeneous.

Note that a representation of L on V is equivalent to a representation on V^* by Remark 48 and $V \cong V^{**}$. As the notion of a derivation extending d_{CE} is dual to the one of a coderivation extending d, we get the following dualized version of Corollary 51.

Proposition 52. *A representation ρ of L on V is equivalent to a derivation $D_{CE}\colon \mathcal{S}(L[1]^*) \otimes V \to \mathcal{S}(L[1]^*) \otimes V$ extending d_{CE} with $D_{CE}^2 = 0$. Explicitly, we have $D_{CE} = -D^*$, where D is the coderivation extending d induced by the dual representation $\rho^\vee$.*

For a fixed representation ρ of L on V, we can then see $\mathcal{S}(L[1]^*) \otimes V$ as our generalized Chevalley–Eilenberg complex with coboundary operator D_{CE}.

4.4 A dead-end

This not only provides us with an explicit construction of the coboundary operator from a given representation, but also gives it the additional structure of a derivation extending d_{CE}. Unfortunately, this came at the cost of the finiteness assumptions we imposed on L on V at the beginning of Section 4.3. As our goal is to establish a generalisation of Theorem 32 – which does not need such assumptions – in terms of L_∞-algebra cohomology, this is not the appropriate framework for our purposes. We can, however, make the following observation.

Remark 53. With our finiteness assumptions on L and V, we have $\mathcal{S}(L[1]^*) \otimes V \cong \mathrm{Hom}(\mathcal{S}(L[1]), V)$, where $\xi \otimes v \in \mathcal{S}(L[1]^*) \otimes V$ is identified with the linear map $\mathcal{S}(L[1]) \to V$, $x \mapsto (-1)^{|x||v|}\xi(x) \cdot v$. For $f \in \mathrm{Hom}(\mathcal{S}(L[1]), V)$ homogeneous, one finds that $D_{CE}f$ is then given by

$$D_{CE}f = \rho(\mathrm{id}_S \otimes f)\Delta_S - (-1)^{|f|} f \circ d. \tag{38}$$

One could then simply define $D_{CE}\colon\ \mathrm{Hom}(\mathcal{S}(L[1]), V) \to \mathrm{Hom}(\mathcal{S}(L[1]), V)$ by (38), even if L and V do not meet our finiteness assumptions. Although there is a priori no reason for $D_{CE}^2 = 0$ to hold in the general case, a straightforward computation shows that it actually does. While this leaves us with nothing but the formula (38) to work with, it also suggests that there should be another approach to L_∞-algebra cohomolgy that gets by without the need of finiteness assumptions.

In [4], the L_∞-algebra cohomology with values in the adjoint representation was introduced in terms of the commutator bracket of coderivations and the isomorphism $\mathrm{Coder}(\overline{\mathcal{S}}(L[1]), \overline{\mathcal{S}}(L[1])) \cong \mathrm{Hom}(\overline{\mathcal{S}}(L[1]), L[1])$. In the next section, we extend this approach to arbitrary representations, which leads to a generalisation of Theorem 32 in a rather natural way.

5 L_∞-algebra cohomology

5.1 The Lie bracket on $\mathrm{Hom}(\overline{\mathcal{S}}(L[1] \oplus V), L[1] \oplus V)$

Recall from Proposition 21 that $\mathrm{Coder}(\mathcal{S}(L[1]))$ is closed under the graded commutator. Together with Theorem 22, this induces a Lie bracket on $\mathrm{Hom}(\mathcal{S}(L[1]), L[1])$. Its explicit formula is

$$[f, g] = f \circ \mu_S(g \otimes \mathrm{id}_S)\Delta_S - (-1)^{|f||g|} g \circ \mu_S(f \otimes \mathrm{id}_S)\Delta_S \tag{39}$$

for $f, g \in \mathrm{Hom}(\mathcal{S}(L[1]), L[1])$ homogeneous.

As L_∞-structures correspond to codifferentials with $d(1) = 0$ and elements in $\mathrm{Hom}(\overline{\mathcal{S}}(L[1]), L[1])$, it is only natural to restrict ourselves to the Lie subalgebra $\mathrm{Hom}(\overline{\mathcal{S}}(L[1]), L[1])$. Keeping the $\mathrm{Hom}(\Bbbk, L[1])$ part corresponds to the framework of *curved L_∞-algebras*, which are L_∞-algebras that also allow for a 0-ary bracket $\Bbbk \to L[1]$.

Remark 54. The same construction also makes $\mathrm{Hom}(\overline{\mathcal{S}}(L[1] \oplus V), L[1] \oplus V)$ into a graded Lie algebra. The decomposition $\mathcal{S}(L[1] \oplus V) \cong \mathcal{S}(L[1]) \otimes \mathcal{S}(V)$ implies that

$$\overline{\mathcal{S}}(L[1] \oplus V) \cong \overline{\mathcal{S}}(L[1]) \otimes \overline{\mathcal{S}}(V) \oplus \overline{\mathcal{S}}(L[1]) \oplus \overline{\mathcal{S}}(V). \tag{40}$$

We can then consider spaces like $\mathrm{Hom}(\overline{\mathcal{S}}(L[1]), L[1])$ and $\mathrm{Hom}(\overline{\mathcal{S}}(L[1]) \otimes V, V)$ as subspaces of $\mathrm{Hom}(\overline{\mathcal{S}}(L[1] \oplus V),\ L[1] \oplus V)$ in the obvious way. The inclusion of $\mathrm{Hom}(\overline{\mathcal{S}}(L[1]), L[1])$ into $\mathrm{Hom}(\overline{\mathcal{S}}(L[1] \oplus V), L[1] \oplus V)$ is then easily seen to preserve the Lie bracket.

Remark 55. In terms of the Lie bracket on $\mathrm{Hom}(\overline{\mathcal{S}}(L[1]), L[1])$, the condition (30) for a linear map $\lambda\colon \overline{\mathcal{S}}(L[1]) \to L[1]$ of degree one to define an L_∞-algebra structure on $L[1]$ becomes

$$\frac{1}{2}[\lambda, \lambda] = 0. \tag{41}$$

By Example 12 and Remark 54, this makes $\mathrm{Hom}(\overline{\mathcal{S}}(L[1] \oplus V), L[1] \oplus V)$ into a DGLA. Solutions of the Maurer–Cartan equation then induce new L_∞-structures on $L[1] \oplus V$ by Example 14.

By abuse of notation, we now denote the (co)products on $\mathcal{S}(L[1])$ and $\mathcal{S}(L[1] \oplus V)$ both by μ_S and Δ_S. This is justified, as they coincide on $\mathcal{S}(L[1]) \subset \mathcal{S}(L[1] \oplus V)$.

In (38), $d = \mu_S(\lambda \otimes \mathrm{id}_S)\Delta_S$ and $\mu_S(\mathrm{id}_S \otimes f)\Delta_S = \mu_S(f \otimes \mathrm{id}_S)\Delta_S$ due to $\mathcal{S}(L[1])$ being (co)commutative. The similarity between (38) and (39) suggests to approach L_∞-algebra cohomology using the Lie bracket on $\mathrm{Hom}(\overline{\mathcal{S}}(L[1] \oplus V), L[1] \oplus V)$.

Proposition 56. *Let $\rho \in \mathrm{Hom}(\mathcal{S}(L[1]) \otimes V, V)$ be of degree one. Then ρ is a representation of L on V if and only if*

$$\rho \circ \mu_S(\rho \otimes \mathrm{id}_S)\Delta_S + \rho \circ \mu_S(\lambda \otimes \mathrm{id}_S)\Delta_S = 0, \tag{42}$$

where λ and ρ are considered as elements of $\mathrm{Hom}(\overline{\mathcal{S}}(L[1] \oplus V), L[1] \oplus V)$.

Proof. Note that $\rho \circ \mu_S(\rho \otimes \mathrm{id}_S)\Delta_S$ and $\rho \circ \mu_S(\rho \otimes \mathrm{id}_S)\Delta_S$ are only possibly nonzero on $\mathcal{S}(L[1]) \otimes V$. For $x_1, \dots, x_{n-1} \in L[1]$ and $x_n \in V$, a routine computation using Lemma 24 shows that

$$\begin{aligned}(\rho \circ \mu_S(\lambda \otimes \mathrm{id}_S)\Delta_S)(x_1 \vee \ldots \vee x_n) \\ = \rho(d(x_1 \vee \ldots \vee x_{n-1}), x_n), \\ (\rho \circ \mu_S(\mathrm{id}_S \otimes \rho)\Delta_S)(x_1 \vee \ldots \vee x_n) \\ = \rho(\mathrm{id}_S \otimes \rho)(\Delta_S(x_1 \vee \ldots \vee x_{n-1}), x_n).\end{aligned}$$

As again $\mu_S(\rho \otimes \mathrm{id}_S)\Delta_S = \mu_S(\mathrm{id}_S \otimes \rho)\Delta_S$ by (co)commutativity of $\overline{\mathcal{S}}(L[1] \oplus V)$, ρ satisfies (33) if and only if it satisfies (42).

Corollary 57. *An element $\rho \in \mathrm{Hom}(\mathcal{S}(L[1]) \otimes V, V)$ of degree one is representation of L on V if and only if $(L[1] \oplus V, \lambda + \rho)$ is an L_∞-algebra.*

Proof. We have

$$\begin{aligned}\frac{1}{2}[\lambda + \rho, \lambda + \rho] &= \frac{1}{2}[\lambda, \lambda] + [\lambda, \rho] + \frac{1}{2}[\rho, \rho] \\ &= \rho \circ \mu_S(\lambda \otimes \mathrm{id}_S)\Delta_S + \rho \circ \mu_S(\rho \otimes \mathrm{id}_S)\Delta_S.\end{aligned}$$

Corollary 58. *The subspace $\mathrm{Hom}(\mathcal{S}(L[1]) \otimes V, V)$ is invariant under the Lie bracket $[\cdot, \cdot]$ and the differential $[\lambda, \cdot]$. Representations (up to homotopy) of L on V are then exactly the Maurer–Cartan elements in $\mathrm{Hom}(\mathcal{S}(L[1]) \otimes V, V)$.*

By applying Proposition 33 to Corollary 57 and using that a representation on V is equivalent to one on $V[1]$ by Remark 48, we obtain the following.

Proposition 59. *A representation of L on V is equivalent to a system of linear maps $\rho_k \colon \bigwedge^{k-1} L \otimes V \to V$ of degree $2 - k$ for $k \geq 1$ such that $\{l_k + \rho_k \colon \bigwedge^k (L \oplus V) \to L \oplus V \mid 1 \leq k < \infty\}$ is an L_∞-structure on $L \oplus V$.*

Remark 60. It is easy to see that the generalized Jacobi identity (27) for $\{l_k + \rho_k \mid 1 \leq k < \infty\}$ has only to be checked on $\bigwedge L^{n-1} \otimes V$ for each $n \geq 1$. Representations of L_∞-algebras are often defined in terms of these equations, see for example ([8], Definition 5.1) and ([3], Definition 18). Similarly, equation (42) on $\mathcal{S}(L[1]) \otimes V$ is easily seen to be the condition imposed on ρ in ([3], Definition 19).

For a fixed representation ρ of L on V, $[\lambda + \rho, \cdot]$ makes $\mathrm{Hom}(\overline{\mathcal{S}}(L[1] \oplus V), L[1] \oplus V)$ into a DGLA. The space $\mathrm{Hom}(\overline{\mathcal{S}}(L[1]), V)$ is then an abelian Lie subalgebra that is invariant under $[\lambda + \rho, \cdot]$. Explicitly, we have for $f \in \mathrm{Hom}(\overline{\mathcal{S}}(L[1]), V)$ homogeneous

$$[\lambda + \rho, f] = \rho(\mathrm{id}_S \otimes f)\Delta_S - (-1)^{|f|} f \circ d. \tag{43}$$

Definition 61. The map $\delta := [\lambda + \rho, \cdot] \colon \mathrm{Hom}(\overline{\mathcal{S}}(L[1]), V) \to \mathrm{Hom}(\overline{\mathcal{S}}(L[1]), V)$ is called the *L_∞-coboundary operator*. The cohomology of the cochain complex $(\mathrm{Hom}(\overline{\mathcal{S}}(L[1]), V), \delta)$ is called the *L_∞-algebra cohomology with values in V*.

Remark 62. For L and V as in Section 4.3, we clearly have $\delta = D_{CE}$. If $L = \mathfrak{g}$ and V are concentrated in degree zero, the décalage isomorphism (5) implies that

$$\begin{aligned}\mathrm{Hom}_p(\overline{\mathcal{S}}(\mathfrak{g}[1]), V) &\cong \prod_{n \geq 1} \mathrm{Hom}_{p-n}(\bigwedge^n \mathfrak{g}, V) \\ &\cong \mathrm{Hom}(\bigwedge^p \mathfrak{g}, V)\end{aligned}$$

for all $p \geq 1$. This way, we recover the usual Lie algebra cohomology.

Example 63 (The adjoint representation)**.** The *adjoint representation* of L on $L[1]$ is given by $\mathcal{S}(L[1]) \otimes L[1] \to L[1]$, $x \otimes y \mapsto \lambda(x \vee y)$. While there are now two distinct copies of $L[1]$ involved, it is evident by (43) that $\delta = [\lambda, \cdot]$, the bracket being the one on $\mathrm{Hom}(\overline{\mathcal{S}}(L[1]), L[1])$. This is the case discussed in [4].

5.2 L_∞-structures induced by 2-cocycles

The description of L_∞-structures, representations (up to homotopy) and the L_∞-coboundary operator all by the same Lie bracket yields the following generalisation of Theorem 32.

Theorem 64. *Let L and V be graded vector spaces and $\lambda \in \mathrm{Hom}(\overline{\mathcal{S}}(L[1]), L[1])$, $\rho \in \mathrm{Hom}(\mathcal{S}(L[1]) \otimes V, V)$ and $\omega \in \mathrm{Hom}(\overline{\mathcal{S}}(L[1]), V)$ be all of degree one. Then $(L[1] \oplus V, \lambda + \rho + \omega)$ is an L_∞-algebra if and only if $(L[1], \lambda)$ is an L_∞-algebra, ρ is a representation of L on V and ω is a V-valued cocycle.*

Proof. The map $\frac{1}{2}[\lambda + \rho + \omega, \lambda + \rho + \omega] = \frac{1}{2}[\lambda, \lambda] + [\lambda, \rho] + \frac{1}{2}[\rho, \rho] + [\lambda + \rho, \omega]$ decomposes itself into linear maps

$$\begin{aligned}\frac{1}{2}[\lambda, \lambda] &\colon \mathcal{S}(L[1]) \to L[1], \\ [\lambda, \rho] + \frac{1}{2}[\rho, \rho] &\colon \mathcal{S}(L[1]) \otimes V \to V, \\ [\lambda + \rho, \omega] &\colon \mathcal{S}(L[1]) \to V.\end{aligned}$$

The assertion then follows from Remark 55, Corollary 58 and the definition of δ.

In terms of antisymmetric brackets, Theorem 64 characterises L_∞-structures on $L \oplus V$ in which for each $n \in \mathbb{N}$, the n-ary bracket decomposes into linear maps

$$\begin{aligned}\bigwedge^n L &\to L, \\ \bigwedge^{n-1} L \otimes V &\to V, \\ \bigwedge^n L &\to V.\end{aligned}$$

These then correspond to cocycles in $\mathrm{Hom}_1(\mathcal{S}(L[1]), V[1]) \cong \mathrm{Hom}_2(\mathcal{S}(L[1]), V)$. So, it is the 2-cocycles that characterise these L_∞-structures, as in the Lie algebra case (cf. [9], Proposition 7.5.18, p. 202).

$$
\begin{aligned}
[\lambda_2, \lambda_m](x \vee y) &= \big(\lambda_2 \circ \mu_S(\lambda_m \otimes \mathrm{id}_S)\Delta_S\big)(x \vee y) + (-1)^{|x|}\lambda_m(x \vee d_2(y)) \\
&= \lambda_2 \circ \mu_S(\delta(x) \otimes \mathrm{id}_S)\Delta_S(y) + (-1)^{|x|}\delta(x)(d_2(y)) \\
&= [\lambda_2, \delta(x)](y), \\
\frac{1}{2}[\lambda_m, \lambda_m](x \vee y) &= \sum (-1)^{|x_{(2)}||y_{(1)}|}\lambda_m(\lambda_m(x_{(1)} \vee y_{(1)}) \vee x_{(2)} \vee y_{(2)}) \\
&= \sum (-1)^{|x_{(2)}|+|x_{(1)}||x_{(2)}|}\delta(x_{(2)})(\delta(x_{(1)} \vee y_{(1)}) \vee y_{(2)}) \\
&= \frac{1}{2}\sum \, (-1)^{x_{(1)}}\delta(x_{(1)})(\delta(x_{(2)} \vee y_{(1)}) \vee y_{(2)}) \\
&\quad + (-1)^{|x_{(2)}|+|x_{(1)}||x_{(2)}|}\delta(x_{(2)})(\delta(x_{(1)} \vee y_{(1)}) \vee y_{(2)}) \\
&= \left(\frac{1}{2}[\cdot,\cdot] \circ (\delta \otimes \delta)\Delta_S\right)(x)(y),
\end{aligned}
$$

5.3 Extensions of L_∞-algebras

We conclude with a brief discussion of extensions of L_∞-algebras. This puts some constructions we discussed in context. The notions are completely analog to the Lie algebra case, see for example ([9], Sections 5.1.3 and 7.5.2).

A graded subspace $I \subset L$ of an L_∞-algebra $(L[1], \lambda)$ is called an *ideal* if $\lambda(x \vee y) \in I[1]$ for all $x \in I[1]$ and $y \in \mathcal{S}(L[1])$. Then L/I carries a canonical L_∞-structure such that the projection $L \to L/I$ is a strict homomorphism of L_∞-algebras. An ideal $I \subset L$ is always an L_∞-subalgebra as in particular $\lambda(x) \in I[1]$ for all $x \in \overline{\mathcal{S}}(I[1])$.

Definition 65. An *extension* of an L_∞-algebra $(L_1[1], \lambda_1)$ by another L_∞-algebra $(L_2[1], \lambda_2)$ is an exact sequence of L_∞-algebras and strict homomorphisms

$$0 \to L_2 \xrightarrow{\iota} L \xrightarrow{p} L_1 \to 0. \tag{44}$$

Given such an exact sequence (44), the graded subspace $L_2 \cong \ker(p) \subset L$ is an ideal and p induces a strict isomorphism $L/L_2 \cong L_1$ of L_∞-algebras.

We then always have $L \cong L_1 \oplus L_2$ (non-canonically) as graded vector spaces, so we are essentially concerned with L_∞-structures on $L_1 \oplus L_2$ such that the canonical maps $L_2 \to L_1 \oplus L_2$ and $L_1 \oplus L_2 \to L_1$ are strict L_∞-algebra homomorphisms. With the decomposition (40), we can decompose such an L_∞-structure $\lambda\colon \overline{\mathcal{S}}((L_1 \oplus L_2)[1]) \to (L_1 \oplus L_2)[1]$ into linear degree one maps

$$
\begin{aligned}
\lambda_1\colon \overline{\mathcal{S}}(L_1[1]) &\to L_1[1], \quad \omega\colon \overline{\mathcal{S}}(L_1[1]) \to L_2[1], \\
0\colon \overline{\mathcal{S}}(L_2[1]) &\to L_1[1], \quad \lambda_2\colon \overline{\mathcal{S}}(L_2[1]) \to L_2[1], \\
0\colon \overline{\mathcal{S}}(L_1[1]) \otimes \overline{\mathcal{S}}(L_2[1]) &\to L_1[1], \\
\lambda_m\colon \overline{\mathcal{S}}(L_1[1]) \otimes \overline{\mathcal{S}}(L_2[1]) &\to L_2[1].
\end{aligned}
$$

5.3.1 Abelian and central extensions

An L_∞-algebra L is called *abelian* if only its 1-ary bracket is nontrivial. An abelian L_∞-algebra is then nothing else than a DG vector space.

An L_∞-algebra extension $L_2 \to L \to L_1$ is called *abelian* if L_2 is abelian. The L_∞-structures constructed in Theorem 64 are examples of abelian extensions of L by V.

Similarly, an extension $L_2 \to L \to L_1$ is called *central* if $\lambda(x \vee y) = 0$ for $x \in L_2[1]$, $y \in \overline{\mathcal{S}}(L[1])$. It is immediate that this is the case if and only if L_2 is abelian and $\lambda_m = 0$. For abelian L_2, the central extensions $L_2 \to L_1 \oplus L_2 \to L_1$ are by Theorem 64 characterised by 2-cocycles of L_1 with values in the trivial representation of L_1 on L_2.

5.3.2 Semidirect sums

An L_∞-algebra $((L_1 \oplus L_2)[1], \lambda)$ is said to be a *semidirect sum* of the L_∞-algebras $(L_1[1], \lambda_1)$ and $(L_2[1], \lambda_2)$ if the canonical sequence $L_2 \to L_1 \oplus L_2 \to L_1$ is an L_∞-algebra extension and if the canonical map $L_1 \to L_1 \oplus L_2$ is a strict homomorphism of L_∞-algebras. This is clearly the case if and only if $\omega = 0$ in the decomposition above. A semidirect sum of L_1 and L_2 is therefore characterised by λ_m. Note that $L_1 \subset L_1 \oplus L_2$ is an ideal if and only if $\lambda_m = 0$. In this case, $L_1 \oplus L_2$ carries the L_∞-structure $\lambda_1 + \lambda_2$ and is called the *direct sum of L_1 and L_2*.

For an arbitrary $\lambda_m \in \mathrm{Hom}_1(\overline{\mathcal{S}}(L_1[1]) \otimes \overline{\mathcal{S}}(L_2[1]), L_2[1])$, the condition for $\lambda_1 + \lambda_2 + \lambda_m$ to define an L_∞-structure on $L_1 \oplus L_2$ becomes

$$[\lambda_1 + \lambda_2, \lambda_m] + \frac{1}{2}[\lambda_m, \lambda_m] = 0. \tag{45}$$

The isomorphism $\mathrm{Hom}(\overline{\mathcal{S}}(L_1[1]) \otimes \overline{\mathcal{S}}(L_2[1]), L_2[1]) \cong \mathrm{Hom}(\overline{\mathcal{S}}(L_1[1]), \mathrm{Hom}(\overline{\mathcal{S}}(L_2[1]), L_2[1]))$ allows for the following characterisation of semidirect sums.

Theorem 66. *Let $\lambda_m \in \mathrm{Hom}(\overline{\mathcal{S}}(L_1[1]) \otimes \overline{\mathcal{S}}(L_2[1]), L_2[1])$ be of degree one. Then λ_m satisfies (45) if and only if the corresponding linear degree one map $\delta\colon \overline{\mathcal{S}}(L_1[1]) \to \mathrm{Hom}(\overline{\mathcal{S}}(L_2[1]), L_2[1])$ is a weak homomorphism of L_∞-algebras in the sense that it satisfies (35).*

Proof. Note that $\mathrm{Hom}(\overline{\mathcal{S}}(L_1[1]) \otimes \overline{\mathcal{S}}(L_2[1]), L_2[1])$ is closed under $[\cdot,\cdot]$ and $[\lambda_1 + \lambda_2, \cdot]$. Therefore, (45) has only to be checked on $\overline{\mathcal{S}}(L_1[1]) \otimes \overline{\mathcal{S}}(L_2[1])$. Let d_1 and d_2 denote the codifferentials on $\mathcal{S}(L_1[1])$ and $\mathcal{S}(L_2[1])$, respectively. For $x \in \overline{\mathcal{S}}(L_1[1])$ and $y \in \overline{\mathcal{S}}(L_2[1])$, we then compute

See this equation above

and $[\lambda_1, \lambda_m](x \vee y) = \lambda_m(d_1(x) \vee y) = (\delta \circ d_1)(x)(y)$.

Example 67. The L_∞-structure on $L \oplus V$ induced by a representation of L on V is a semidirect sum. For compliance with Theorem 66, note that $\mathfrak{gl}(L_2[1]) \subset \mathrm{Hom}(\overline{\mathcal{S}}(L_2[1]), L_2[1])$ is a Lie subalgebra.

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